Existence and Uniqueness of L^2-Solutions at Zero-Diffusivity in the Kraichnan Model of a Passive Scalar

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Abstract

We study Kraichnan's model of a turbulent scalar, passively advected by a Gaussian random velocity field delta-correlated in time, for every space dimension $d \geq 2$ and eddy-diffusivity (Richardson) exponent $0 < \zeta < 2$. We prove that at zero molecular diffusivity, or $\kappa = 0$, there exist unique weak solutions in $L^2\left(\Omega^{\otimes N}\right)$ to the singular-elliptic, linear PDE's for the stationary N-point statistical correlation functions, when the scalar field is confined to a bounded domain Ω with Dirichlet b.c. Under those conditions we prove that the N-body elliptic operators in the L^2 spaces have purely discrete, positive spectrum and a minimum eigenvalue of order $L^{-\gamma}$, with $\gamma = 2 - \zeta$ and with L the diameter of Ω. We also prove that the weak L^2-limits of the stationary solutions for positive, pth-order hyperdiffusivities $\kappa_p > 0$, $p \geq 1$, exist when $\kappa_p \to 0$ and coincide with the unique zero-diffusivity solutions. These results follow from a lower estimate on the minimum eigenvalue of the N-particle eddy-diffusivity matrix, which is conjectured for general N and proved in detail for $N = 2, 3, 4$. Some additional issues are discussed: (1) Hölder regularity of the solutions; (2) the reconstruction of an invariant probability measure on scalar fields from the set of N-point correlation functions, and (3) time-dependent weak solutions to the PDE's for N-point correlation functions with L^2 initial data.

1 Introduction

We study the model problem of a scalar field $\theta(\mathbf{r},t)$ satisfying an advection-diffusion equation

$$(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}})\theta = \kappa \triangle_{\mathbf{r}} \theta + f \tag{1}$$

in a bounded domain Ω of Euclidean d-dimensional space $\mathbf{R}^d$, with Dirichlet conditions on the boundary $\partial\Omega$. The scalar source $f(\mathbf{r},t)$ is assumed a Gaussian random field, white-noise in time but regular in space. Precisely, we take f with mean $\langle f(\mathbf{r},t)\rangle = \overline{f}(\mathbf{r}) \in L^2(\Omega)$ and covariance

$$\langle f(\mathbf{r},t)f(\mathbf{r}',t')\rangle - \langle f(\mathbf{r},t)\rangle\langle f(\mathbf{r}',t')\rangle = F(\mathbf{r},\mathbf{r}')\delta(t-t') \tag{2}$$

with $F \in L^2(\Omega\otimes\Omega)$. The velocity field is also assumed Gaussian, white-noise in time, zero-mean with covariance

$$\langle v_i(\mathbf{r},t)v_j(\mathbf{r}',t')\rangle = V_{ij}(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{3}$$

The velocity to be considered is a divergence-free random field in $\mathbf{R}^d$ and, for convenience, statistically homogeneous. There is no reason to insist on Dirichlet b.c. for the velocity field. The spatial covariance matrix $\mathbf{V}$ we consider is defined by the Fourier integral

$$V_{ij}(\mathbf{r}) = D_0 \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left(k^2+m^2\right)^{-(d+\zeta)/2} P_{ij}^{\perp}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{4}$$

where $0 < \zeta < 2$ and $P_{ij}^{\perp}(\mathbf{k})$ is the projection in $\mathbf{R}^d$ onto the subspace perpendicular to $\mathbf{k}$. This automatically defines a suitable positive-definite, symmetric matrix-valued function, divergence-free in each index. The model originates in the 1968 work of R. H. Kraichnan [1] and has been the subject of recent analytical investigations [2, 3, 4, 5, 6, 7, 8]. It is not hard to show that

$$V_{ij}(\mathbf{r}) \sim V_0\delta_{ij} - D_1 \cdot r^{\zeta} \cdot \left[\delta_{ij} + \frac{\zeta}{d-1}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right] + \cdots \tag{5}$$

asymptotically for $mr \ll 1$, with V_0 and D_1 constants proportional to D_0, given below. See also Section 4.1 of [4]. The exponent ζ has the physical interpretation of an "eddy-diffusivity exponent" analogous to the Richardson exponent 4/3 [9].

The remarkable feature of Kraichnan's model, which makes it, in a certain sense, "exactly soluble" is that N-th order correlation functions $\Theta_N(\mathbf{r}_1, ..., \mathbf{r}_N; t) = \langle \theta(\mathbf{r}_1, t) \cdots \theta(\mathbf{r}_N, t) \rangle$ satisfy *closed* equations of the form

$$\partial_t \Theta_N = -\widetilde{\mathcal{H}}_N^{(\kappa)} \Theta_N + \sum_n \overline{f}(\mathbf{r}_n) \Theta_{N-1}(...\widehat{\mathbf{r}_n}...) + \sum_{\text{pairs } \{nm\}} F(\mathbf{r}_n, \mathbf{r}_m) \Theta_{N-2}(...\widehat{\mathbf{r}_n}...\widehat{\mathbf{r}_m}...). \tag{6}$$

In this equation for the N-correlator only itself and lower-order correlators appear [1, 2, 3, 4]. Here $\widetilde{\mathcal{H}}_N^{(\kappa)}$ is an elliptic partial-differential operator in $\Omega^{\otimes N}$ defined as

$$\widetilde{\mathcal{H}}_N^{(\kappa)} = -\frac{1}{2} \sum_{i,j=1}^{d} \sum_{n,m=1}^{N} \frac{\partial}{\partial x_{in}} \left[V_{ij}(\mathbf{r}_n - \mathbf{r}_m) \frac{\partial}{\partial x_{jm}} \cdot \right] - \kappa \sum_{n=1}^{N} \triangle_{\mathbf{r}_n}, \tag{7}$$

with Dirichlet b.c., where x_{in} are Cartesian coordinates in $\left(\mathbf{R}^d\right)^{\otimes N}$. However, the operator $\widetilde{\mathcal{H}}_N$ obtained by taking $\kappa \to 0$ is degenerate, i.e. it is singular-elliptic. We refer to $\widetilde{\mathcal{H}}_N$ as the N-body *convective operator* because it accounts for the effects of the velocity advection alone in the equation (6) for N-point correlations. Because of the degeneracy for $\kappa \to 0$, the solutions of the parabolic equation are expected in that limit to lie only in a Hölder class $C^\gamma\left(\Omega^{\otimes N}\right)$ with $\gamma = 2 - \zeta$. As the differential operator is of second-order, these solutions must then be taken in a suitable weak sense. Despite the degeneracy, the linear operator $\widetilde{\mathcal{H}}_N$ is formally self-adjoint and nonnegative in the L^2 inner product of functions on $\Omega^{\otimes N}$. This suggests that an L^2-theory of weak solutions to Eq.(6) may be appropriate. We shall develop here such a theory in detail. The key to the analysis of the $\kappa \to 0$ limiting solutions is a proof of existence and uniqueness directly for $\kappa = 0$.

Let us state precisely the main theorems of this work. We shall actually consider a somewhat more general model than Eq.(1), namely,

$$(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{r}})\theta = -\kappa_p(-\triangle_{\mathbf{r}})^p \theta + f \tag{8}$$

with $p \geq 1$, in which κ_p is a so-called hyperdiffusivity of order p. This allows us to establish a universality result concerning the independence of limits on p. In this case, the closed correlation

equations (6) are still satisfied, with the operator (7) replaced by

$$\widetilde{\mathcal{H}}_N^{(\kappa_p)} = -\frac{1}{2}\sum_{i,j=1}^{d}\sum_{n,m=1}^{N}\frac{\partial}{\partial x_{in}}\left[V_{ij}(\mathbf{r}_n-\mathbf{r}_m)\frac{\partial}{\partial x_{jm}}\cdot\right] + \kappa_p\sum_{n=1}^{N}(-\triangle_{\mathbf{r}_n})^p. \tag{9}$$

Note that this operator requires higher-order Dirichlet b.c., namely, elements in its domain must have zero trace on the boundary for the first $k = [\![p-(1/2)]\!]$ derivatives. However, our first main result is for the solution of that equation directly at $\kappa_p = 0$:

Theorem 1 *Assume that $d \geq 2$ and $0 < \zeta < 2$. Then, for integers $N \geq 1$, the equation (6) at $\kappa = 0$ has a unique stationary weak solution Θ_N^* in $L^2\left(\Omega^{\otimes N}\right)$. Away from the codimension-d set where pairs of points in $\mathbf{R} = (\mathbf{r}_1, ..., \mathbf{r}_N)$ coincide, the solution $\Theta_N^*(\mathbf{R})$ is in $H_0^1\left(\Omega^{\otimes N}\right)$.*

This ideal zero-diffusivity solution is, in fact, the physically relevant one in the limits $\kappa_p \to 0$, as shown by our second main result:

Theorem 2 *Assume that $d \geq 2$ and $0 < \zeta < 2$, and also $p \geq 1$.*
(i) For integers $N \geq 1$, the equation (6), with $\widetilde{\mathcal{H}}^{(\kappa)}$ generalized to $\widetilde{\mathcal{H}}^{(\kappa_p)}$, has a unique stationary weak solution $\Theta_N^{(\kappa_p)*}$ in $L^2\left(\Omega^{\otimes N}\right)$, which, in fact, belongs to the Sobolev space $H_0^p\left(\Omega^{\otimes N}\right)$.
(ii) The weak-L^2 limit exists as $\kappa_p \to 0$ and $w - \lim_{\kappa_p\to 0}\Theta_N^{(\kappa_p)*} = \Theta_N^*$.

To prove these results requires a spectral analysis of the N-body convective operator $\widetilde{\mathcal{H}}_N$. In fact, we show that this operator has pure point spectrum, using a criterion borrowed from a work of R. T. Lewis [10]. Discreteness of the spectrum was already shown by Majda [3] in his simple version of the model. For our theorems above, we do not really require that $\widetilde{\mathcal{H}}_N$ have a compact inverse, but merely a bounded inverse. To prove this, we require an estimate from below on the quadratic form associated to $\widetilde{\mathcal{H}}_N$. This is proved in two steps. First, for each integer $N \geq 1$ we define the $(Nd)\times(Nd)$-dimensional matrix $[\mathbf{G}_N(\mathbf{R})]_{in,jm} = \langle v_i(\mathbf{r}_n)v_j(\mathbf{r}_m)\rangle,\ i,j = 1,...,d, n,m = 1,...,N$. Physically, this is interpreted as an *N-particle eddy-diffusivity matrix.* Mathematically, it is the nonnegative Gramian matrix of the Nd elements $v_i(\mathbf{r}_n)$ in the L^2 inner-product space of the random velocity field. It is nonsingular if and only if these Nd elements are linearly independent. We shall prove below (Proposition 2) that its minimum

eigenvalue obeys $\lambda_N^{\min}(\mathbf{R}) \geq C_N[\rho(\mathbf{R})]^\zeta$, where $\rho(\mathbf{R}) = \min_{n\neq m}|\mathbf{r}_n - \mathbf{r}_m|$, when $N = 2, 3, 4$. The second step of the proof uses only this property of $\mathbf{G}_N(\mathbf{R})$, which is conjectured to hold for all $N \geq 1$. As a consequence of this estimate, we prove a lower bound on the operator quadratic form, reminiscent of the well-known *Hardy inequality* [11] (Theorem 330). For the operator with Dirichlet b.c. we may adapt a convenient proof of the Hardy-type inequality due also to Lewis [10]. Unfortunately, as explained below, this proof does not work with periodic b.c. although the inequality is likely to hold there as well (for zero-mean functions). Lewis' argument is also too restrictive to permit treatment of other models with more natural b.c. on the velocity field. In a real turbulent flow with velocity field governed by the Navier-Stokes equation, the realizations of the velocity field would satisfy also Dirichlet b.c. This behavior may be mimicked with the Gaussian random velocity fields by taking as their covariance

$$V_{ij}^{(\Omega)}(\mathbf{r}, \mathbf{r}') = \Delta_\Omega(\mathbf{r}) V_{ij}(\mathbf{r} - \mathbf{r}') \Delta_\Omega(\mathbf{r}'), \tag{10}$$

in which $\Delta_\Omega(\mathbf{r})$ is a suitable "wall-damping function". It should be taken as some decreasing function of the distance to the boundary $\partial\Omega$, vanishing there as some power. Of course, with this choice of velocity covariance, a lower bound directly follows from our present work that $\lambda_N^{\min}(\mathbf{R}) \geq C_N[\rho(\mathbf{R})]^\zeta[\Delta_\Omega(\mathbf{R})]^2$, where $\Delta_\Omega(\mathbf{R}) = \min_{1\leq n\leq N}\Delta_\Omega(\mathbf{r}_n)$. While we expect the main results of this work to carry over to such models, it requires a different proof of the generalized Hardy inequality. We will return to this problem in a later work.

Let us summarize the contents of this paper: In Section 2 we establish the required properties of the model velocity covariance and the resulting N-particle eddy-diffusivity matrix, in particular the lower bound on the minimum eigenvalue. In Section 3 we study the operator quadratic form, and prove its principal properties, such as the generalized Hardy inequality. Finally, in Section 4 we exploit these results to prove the main Theorems 1 and 2 above. In the conclusion Section 5 we briefly discuss three other problems: regularity of solutions, the reconstruction of an invariant measure from the stationary N-point correlation functions, and time-dependent solutions to the parabolic PDE's for the N-correlators.

2 Properties of the N-Particle Eddy-Diffusivity Matrix

(2.1) The Velocity Covariance Matrix

We first state and prove the regularity properties of the velocity covariance matrix elements ($V_{ij}(\mathbf{r})$) that we will need for later analysis. We have made the choice of Eq.(4) just for specificity. In fact, any velocity covariance with the following properties would suffice.

Lemma 1 *The elements of velocity covariance matrix $V_{ij}(\mathbf{r})$, $\mathbf{r} \in \mathbf{R}^d$, are C^∞ in $\mathbf{r}$ if $\mathbf{r} \neq 0$, and C^ζ near $\mathbf{r} = \mathbf{0}$, with $\zeta \in (0, 2)$. Moreover, there is a positive number ρ_0 such that if $r \in [0, \rho_0]$, we have the local expansion:*

$$V_{ij}(\mathbf{r}) = V_0\delta_{ij} - D_1 \cdot r^\zeta \cdot \left[\delta_{ij} + \frac{\zeta}{d-1}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right] + O\left(m^2 r^2\right) \tag{11}$$

Proof: The matrix $V_{ij}(\mathbf{r})$ can be written as

$$V_{ij}(\mathbf{r}) = V(r)\delta_{ij} + \partial_i\partial_j W(r), \tag{12}$$

where the function $V(r)$ is defined by the integral

$$V(r) = D_0 \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left(k^2 + m^2\right)^{-(d+\zeta)/2} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{13}$$

and $W(r)$ is given by the (for $d = 2$, principal part) integral

$$W(r) = D_0 \int \frac{d^d\mathbf{k}}{(2\pi)^d} \left(k^2 + m^2\right)^{-(d+\zeta)/2} \frac{1}{k^2} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{14}$$

so that $- \triangle W = V$. The scalar function $V(r)$ is essentially just the standard Bessel potential kernel [12], and may thus be expressed in terms of a modified Bessel function:

$$V(r) = D_0 \frac{2^{1-(\zeta/2)} m^{-\zeta}}{(4\pi)^{d/2}\Gamma\left(\frac{d+\zeta}{2}\right)} \cdot (mr)^{\zeta/2} K_{\zeta/2}(mr). \tag{15}$$

The Hessian $\partial_i\partial_j W(r)$ of the function W of magnitude $r = |\mathbf{r}|$ alone is

$$\partial_i\partial_j W(r) = \delta_{ij} A(r) + \widehat{r}_i \widehat{r}_j \cdot r \frac{dA}{dr}(r), \tag{16}$$

with $A(r) = W'(r)/r$ and $\widehat{\mathbf{r}} = \mathbf{r}/r$. However, because $\mathrm{Tr}\,(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} W) = -V$, a Cauchy-Euler equation follows for $A(r)$:

$$r\frac{dA}{dr}(r) + d \cdot A(r) = -V(r). \tag{17}$$

Due to the rapid decay of its Fourier transform, the function $A(r)$ is continuous. Thus, the relevant solution is found to be

$$A(r) = -r^{-d} \int_0^r \rho^{d-1} V(\rho) d\rho. \tag{18}$$

in terms of $V(r)$. Using this expression for $A(r)$, along with Eq.(16), we thus find

$$V_{ij}(\mathbf{r}) = (V(r) + A(r))\delta_{ij} - (V(r) + d \cdot A(r))\widehat{r}_i\widehat{r}_j, \tag{19}$$

for V_{ij} as a linear functional of V. If V has a power-law form, $V(r) = Br^{\xi}$, then it is easy to calculate that

$$V_{ij}(\mathbf{r}) = Br^{\xi}\frac{d-1}{d+\xi}\left[\delta_{ij} + \frac{\xi}{d-1}\left(\delta_{ij} - \widehat{r}_i\widehat{r}_j\right)\right]. \tag{20}$$

By means of the known Frobenius series expansions for the modified Bessel functions (e.g. [13], (9.6.2),(9.6.10)), it follows that

$$z^{\nu}K_{\nu}(z) = \frac{\Gamma(\nu)}{2^{1-\nu}} - \frac{\Gamma(1-\nu)}{\nu \cdot 2^{1+\nu}} z^{2\nu} + O\left(z^2\right). \tag{21}$$

From these terms for $K_{\nu}(z)$ we obtain, upon substituting Eq.(15) into Eq.(19), the claimed asymptotic expression for $V_{ij}(\mathbf{r})$ in Eq.(11), with

$$V_0 = D_0 \frac{(d-1)\Gamma\left(\frac{\zeta}{2}\right)}{(4\pi)^{d/2} \cdot d \cdot \Gamma\left(\frac{d+\zeta}{2}\right)} \cdot m^{-\zeta}, \tag{22}$$

and

$$D_1 = D_0 \frac{(d-1)\Gamma\left(\frac{2-\zeta}{2}\right)}{(4\pi)^{d/2} \cdot 2^{\zeta} \cdot \zeta \cdot \Gamma\left(\frac{d+\zeta+2}{2}\right)}. \tag{23}$$

Finally, the Bessel function $K_{\nu}(z)$ is analytic in the complex plane with a branch cut along the negative real axis. Thus, the stated smoothness properties of V_{ij} follow. $\square$

We shall denote the second term on the right hand side of (11) as $-r^{\zeta}Q_{ij}$. Obviously, (Q_{ij}) is positive definite uniformly in r. We will denote by $\mathbf{r}_{nm} = \mathbf{r}_n - \mathbf{r}_m$ the vector, and $r_{nm} = |\mathbf{r}_n - \mathbf{r}_m|$

the scalar distance from $\mathbf{r}_n$ to $\mathbf{r}_m$; V_{ij} the matrix elements, and $\mathbf{V}_{nm}$ the matrix evaluated at $\mathbf{r}_{nm}$. We show two more lemmas.

Lemma 2 *Let $\mathbf{r}_i$, $r = 1, 2, 3$, be any three points in R^d, and $r_{12} \le r_{13}$, $r_{12} \le r_{23}$. Then there is a constant $\bar{C}$ depending on ρ_0 and ζ in Lemma 1 but independent of r_{12}, r_{13}, and r_{23} such that:*

$$|\mathbf{V}_{13} - \mathbf{V}_{23}| \le \bar{C} r_{12} \max(r_{13}^{\zeta-1}, r_{23}^{\zeta-1}).$$

Proof: If $\zeta \in (1, 2)$, then $\nabla \mathbf{V} \in C^{\zeta-1}$, and so by Lemma 1:

$$|\mathbf{V}_{13} - \mathbf{V}_{23}| = |\mathbf{r}_{12} \cdot \nabla_{\mathbf{r}_1} \mathbf{V}|_{\mathbf{r}_\theta}| = |\mathbf{r}_{12} \cdot (\nabla_{\mathbf{r}_1} \mathbf{V}|_{\mathbf{r}_\theta} - \nabla_{\mathbf{r}_1} \mathbf{V}|_{\mathbf{r}=0})|,$$

$$\le \bar{c} r_{12} r_\theta^{\zeta-1} \le \bar{c} r_{12} \max(r_{12}^{\zeta-1}, r_{23}^{\zeta-1}),$$

where $\mathbf{r}_\theta = \theta \mathbf{r}_1 + (1-\theta)\mathbf{r}_2$, for some $\theta \in (0, 1)$. The case $\zeta = 1$ is obviously true by the mean value theorem. Now if $\zeta \in (0, 1)$, $\max(r_{13}, r_{23}) \ge \rho_0$, then using $\mathbf{V} \in C^1$ away from zero, we have:

$$|\mathbf{V}_{13} - \mathbf{V}_{23}| \le \bar{c} r_{12} \le \bar{c} r_{12} (m \max(r_{13}, r_{23}))^{\zeta-1}$$

$$\le \bar{c}(\rho_0, m) r_{12} \max(r_{13}^{\zeta-1}, r_{23}^{\zeta-1}).$$

If $\zeta \in (0, 1)$, and $\max(r_{13}, r_{23}) < \rho_0$, we employ local expansion to calculate for any $\mathbf{x} \ne \mathbf{y}$:

$$|V_{ij}(\mathbf{x}) - V_{ij}(\mathbf{y})| \le \bar{c} |(|\mathbf{x}|^\zeta - |\mathbf{y}|^\zeta)[\delta_{ij} + \frac{\zeta}{(d-1)}(\delta_{ij} - \frac{x_i x_j}{x^2})]|$$

$$+ \bar{c} |y^\zeta (\frac{x_i x_j}{x^2} - \frac{y_i y_j}{y^2})|$$

$$\le \bar{c} \max(x^{\zeta-1}, y^{\zeta-1}) |\mathbf{x} - \mathbf{y}| + \bar{c} y^\zeta \left| \frac{x_i x_j y^2 - y_i y_j x^2}{x^2 y^2} \right|.$$

The latter term is just:

$$\bar{c} y^\zeta |\frac{(x_i x_j - y_i y_j) y^2 + y_i y_j (y^2 - x^2)}{x^2 y^2}|$$

$$= \bar{c} y^\zeta \left(\frac{|\mathbf{x} - \mathbf{y}|}{|x|} + \frac{|\mathbf{x} - \mathbf{y}| y}{x^2} + \frac{|\mathbf{x} - \mathbf{y}|(x + y)}{x^2} \right).$$

With no loss of generality, we assume that $y \leq x$; otherwise, we simply switch $\mathbf{x}$ and $\mathbf{y}$. It follows that

$$|V_{ij}(\mathbf{x}) - V_{ij}(\mathbf{y})| \leq \bar{c}|\mathbf{x}-\mathbf{y}| \max(x^{\zeta-1}, y^{\zeta-1}) + \bar{c}|\mathbf{x}-\mathbf{y}|x^{\zeta-1} \leq \bar{c}|\mathbf{x}-\mathbf{y}| \max(x^{\zeta-1}, y^{\zeta-1}).$$

We complete the proof with $\mathbf{x} = \mathbf{r}_{13}$, and $\mathbf{y} = \mathbf{r}_{23}$.

Lemma 3 *Assume that* $r_{12} \leq r_{34}$; $r_{13} = O(r_{14}) = O(r_{23}) = O(r_{24})$; $\frac{r_{34}}{r_{13}} \leq \epsilon \in (0,1)$. *Then there exist* ϵ_0 *and a positive constant* $\bar{c}_1$ *depending on* ρ_0, ζ, *maximum and minimum ratios of* r_{13}, r_{14}, r_{23}, *and* r_{24}, *such that:*

$$|\mathbf{V}_{13} - \mathbf{V}_{14} - (\mathbf{V}_{23} - \mathbf{V}_{24})| \leq \bar{c}_1 r_{12} r_{34} r_{13}^{\zeta-2},$$

for all $\epsilon \in (0, \epsilon_0)$.

Proof: Applying the mean value theorem to $F(\mathbf{r}_1) \equiv \mathbf{V}_{13} - \mathbf{V}_{14}$, we get for $\mathbf{r}_\theta = \theta\mathbf{r}_1 + (1-\theta)\mathbf{r}_2$ that:

$$F(\mathbf{r}_1) - F(\mathbf{r}_2) = \mathbf{r}_{12} \cdot \nabla_{\mathbf{r}_1} F|_{\mathbf{r}_\theta}.$$

If $\max(r_{13}, r_{24}) \geq \frac{\rho_0}{2}$, then

$$\nabla_{\mathbf{r}_1} F|_{\mathbf{r}_\theta} = \nabla_{\mathbf{r}_1} \mathbf{V}_{13} - \nabla_{\mathbf{r}_1} \mathbf{V}_{14}|_{\mathbf{r}_1 = \mathbf{r}_\theta}.$$

By the smoothness of $\nabla_{\mathbf{r}_1} \mathbf{V}_{1i}$ when the distance of $\mathbf{r}_1$ from $\mathbf{r}_i$, $i = 3, 4$ is larger than $\frac{\rho_0}{4}$ (which is possible if ϵ is small enough), we obtain:

$$|\nabla_{\mathbf{r}_1} F|_{\mathbf{r}=\mathbf{r}_\theta}| \leq \bar{c}_1 \rho_0 r_{34},$$

from which it follows that:

$$|F(\mathbf{r}_1) - F(\mathbf{r}_2)| \leq \bar{c}_1 r_{12} r_{34} \leq \bar{c}_1 r_{12} r_{34} r_{13}^{\zeta-2}.$$

On the other hand, if $\max(r_{13}, r_{24}) < \frac{\rho_0}{2}$, we use local expansion in Lemma 1 to get for each matrix element:

$$(F(\mathbf{r}_1) - F(\mathbf{r}_3))_{ij} \quad = \quad (-D_1) r_{13}^{\zeta} [\delta_{ij} + \frac{\zeta}{d-1}(\delta_{ij} - \frac{\mathbf{r}_{13}^{(i)} \mathbf{r}_{13}^{(j)}}{r_{13}^2})]$$

$$
\begin{aligned}
&+ \quad D_1 r_{14}^{\zeta}[\delta_{ij} + \frac{\zeta}{d-1}(\delta_{ij} - \frac{\mathbf{r}_{14}^{(i)}\mathbf{r}_{14}^{(j)}}{r_{14}^2})] \\
&- \quad (1 \to 2) \\
&= \quad (-D_1)\mathbf{r}_{12} \cdot \nabla_{\mathbf{r}_1}(r_{13}^{\zeta}[\delta_{ij} + \frac{\zeta}{d-1}(\delta_{ij} - \frac{\mathbf{r}_{13}^{(i)}\mathbf{r}_{13}^{(j)}}{r_{13}^2})] \\
&- \quad r_{14}^{\zeta}[\delta_{ij} + \frac{\zeta}{d-1}(\delta_{ij} - \frac{\mathbf{r}_{14}^{(i)}\mathbf{r}_{14}^{(j)}}{r_{14}^2})])(\mathbf{r}_1 = \mathbf{r}_\theta),
\end{aligned}
\tag{24}
$$

where the notation $(1 \to 2)$ means the same terms as before except that subscript 1 is replaced by 2. Let us calculate the $\mathbf{r}_1$ gradient in (24) as (k meaning the kth component of this gradient):

$$
\zeta r_{13}^{\zeta-1}\frac{\mathbf{r}_1^{(k)} - \mathbf{r}_3^{(k)}}{r_{13}}[\delta_{ij} + \frac{\zeta}{d-1}(\delta_{ij} - \frac{\mathbf{r}_{13}^{(i)}\mathbf{r}_{13}^{(j)}}{r_{13}^2})] + r_{13}^{\zeta}\frac{-\zeta}{d-1} \cdot \nabla_{\mathbf{r}_1}\frac{\mathbf{r}_{13}^{(i)}\mathbf{r}_{13}^{(j)}}{r_{13}^2} - (3 \to 4)
$$

$$
\begin{aligned}
&= \quad \zeta(r_{13}^{\zeta-1} - r_{14}^{\zeta-1})\frac{\mathbf{r}_1^{(k)} - \mathbf{r}_3^{(k)}}{r_{13}}[\delta_{ij} + \frac{\zeta}{d-1}(\delta_{ij} - \frac{\mathbf{r}_{13}^{(i)}\mathbf{r}_{13}^{(j)}}{r_{13}^2})] \\
&+ \quad \zeta r_{14}^{\zeta-1}\left(\frac{\mathbf{r}_1^{(k)} - \mathbf{r}_3^{(k)}}{r_{13}}[\delta_{ij} + \frac{\zeta}{d-1}(\delta_{ij} - \frac{\mathbf{r}_{13}^{(i)}\mathbf{r}_{13}^{(j)}}{r_{13}^2})] - (3 \to 4)\right) \\
&+ \quad \frac{-\zeta}{d-1}\left(r_{13}^{\zeta}\frac{-2\mathbf{r}_{13}^{(i)}\mathbf{r}_{13}^{(j)}\mathbf{r}_{13}^{(k)}}{r_{13}^4} + r_{13}^{\zeta}\frac{\delta_{ik}\mathbf{r}_{13}^{(j)}}{r_{13}^2} + r_{13}^{\zeta}\frac{\mathbf{r}_{13}^{(i)}\delta_{jk}}{r_{13}^2} - (3 \to 4)\right).
\end{aligned}
\tag{25}
$$

Note that the first term of the right hand side of (25) is bounded by:

$$
C(\zeta, d)|r_{13}^{\zeta-1} - r_{14}^{\zeta-1}| \leq C(\zeta, d) r_{23}^{\zeta-2} r_{34}.
$$

We can think of

$$
\frac{\mathbf{r}_1^{(k)} - \mathbf{r}_3^{(k)}}{r_{13}}[\delta_{ij} + \frac{\zeta}{d-1}(\delta_{ij} - \frac{\mathbf{r}_{13}^{(i)}\mathbf{r}_{13}^{(j)}}{r_{13}^2})]
$$

as a bounded C^1 function of the unit vector $\hat{\mathbf{r}}_{13}$ along $\mathbf{r}_{13}$. Hence the second term of (25) being the difference of two values of this function at two points $\hat{\mathbf{r}}_{13}$ and $\hat{\mathbf{r}}_{14}$ is of the order $O(\frac{r_{34}}{r_{13}})$. Thus the second term is bounded by

$$
\bar{c}_1 r_{14}^{\zeta-1} r_{34} r_{13}^{-1} \leq \bar{c}_1 r_{13}^{\zeta-2} r_{34}.
$$

Similarly, the third term is bounded as such. Combining the above with (24) we deduce that $|F(\mathbf{r}_1) - F(\mathbf{r}_3)| \leq \bar{c}_1 r_{12} r_{34} r_{13}^{\zeta-2}$. The proof of the lemma is complete.

(2.2) The N-Point Eddy-Diffusivity (Gramian) Matrix

As in the Introduction, we define for each integer $N \geq 1$ the $(Nd) \times (Nd)$-dimensional Gramian matrix $[\mathbf{G}_N(\mathbf{R})]_{in,jm} = \langle v_i(\mathbf{r}_n) v_j(\mathbf{r}_m) \rangle$. For the moment we consider general velocity covariances, given by a Fourier integral

$$V_{ij}(\mathbf{r}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, \widehat{V}_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{26}$$

with $\widehat{\mathbf{V}}(\mathbf{k}) \geq \mathbf{0}$ for each $\mathbf{k} \in \mathbf{R}^d$. The basic properties are contained in:

Proposition 1 *For each $N \geq 2$ the matrix $\mathbf{G}_N(\mathbf{R})$ has the following properties:*

(i) $\mathbf{G}_N(\mathbf{R}) \geq \mathbf{0}$.

(ii) Assume that for all $\mathbf{k} \in \mathbf{R}^d$ the velocity spectral matrix $\widehat{\mathbf{V}}(\mathbf{k}) > \mathbf{0}$ on the subspace orthogonal to the vector $\mathbf{k}$. In that case, $\mathbf{G}_N(\mathbf{R})$ has a nontrivial null space if and only if $\mathbf{r}_n = \mathbf{r}_m$ for some pair of points $n \neq m$.

(iii) For the same hypothesis as (ii), if $\{\mathbf{r}_1, ..., \mathbf{r}_N\}$ has K subsets of coinciding points, with N_k points in the kth subset, $k = 1, ..., K$, then the dimension of the null space of $\mathbf{G}_N(\mathbf{R})$ is $\sum_{k=1}^{K}(N_k - 1)d$. The null space consists precisely of vectors $\mathbf{\Xi} = (\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_N)$ with the property that

$$\sum_{n_k=1}^{N_k} \boldsymbol{\xi}_{n_k} = \mathbf{0}, \tag{27}$$

for each $k = 1, ..., K$, where the sum runs over the N_k coinciding points in the kth subset.

Proof: *(i)* Obvious from the stochastic representation. *(ii)*& *(iii)* Let us assume that the Nd-dimensional vector $\mathbf{\Xi} = (\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_N)$ belongs to $\mathrm{Ker}\mathbf{G}_N(\mathbf{R})$. Then, using the definition of $\mathbf{G}_N(\mathbf{R})$ and the Fourier integral representation Eq.(26), it follows that

$$0 = \langle \mathbf{\Xi}, \mathbf{G}_N(\mathbf{R})\mathbf{\Xi} \rangle = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \overline{\left(\sum_{n=1}^{N} \boldsymbol{\xi}_n e^{i\mathbf{k}\cdot\mathbf{r}_n} \right)} \cdot \widehat{\mathbf{V}}(\mathbf{k}) \cdot \left(\sum_{n=1}^{N} \boldsymbol{\xi}_n e^{i\mathbf{k}\cdot\mathbf{r}_n} \right). \tag{28}$$

This can only occur if the nonnegative integrand vanishes for a.e. $\mathbf{k} \in \mathbf{R}^d$. Because of our assumption on $\widehat{\mathbf{V}}(\mathbf{k})$, this implies that

$$\sum_{n=1}^{N} \boldsymbol{\xi}_n e^{i\mathbf{k}\cdot\mathbf{r}_n} = \alpha(\mathbf{k}) \cdot \mathbf{k}, \tag{29}$$

for a.e. $\mathbf{k} \in \mathbf{R}^d$ with some complex coefficient $\alpha(\mathbf{k})$. Taking the vector cross product with respect to $\mathbf{k}$ and then Fourier transforming, we obtain that

$$\sum_{n=1}^{N} \boldsymbol{\xi}_n \times \boldsymbol{\nabla}\delta(\mathbf{r}-\mathbf{r}_n) = \mathbf{0}, \tag{30}$$

in the sense of distributions. Therefore, for any smooth test function φ,

$$\sum_{n=1}^{N} \boldsymbol{\xi}_n \times (\boldsymbol{\nabla}\varphi)(\mathbf{r}_n) = \mathbf{0}. \tag{31}$$

Because the values of $\boldsymbol{\nabla}\varphi$ may be arbitrarily specified at any set of distinct points, it follows that

$$\sum_{k=1}^{K} \left(\sum_{n_k=1}^{N_k} \boldsymbol{\xi}_{n_k} \right) \times \mathbf{a}_k = \mathbf{0} \tag{32}$$

with $\mathbf{a}_k \in \mathbf{R}^d$ arbitrary. This immediately implies that Eq.(27) is both necessary and sufficient for $\boldsymbol{\Xi}$ to belong to $\mathrm{Ker}\mathbf{G}_N(\mathbf{R})$. Furthermore, this subspace has dimension $\sum_{k=1}^{K}(N_k-1)d$, which completes the proof of *(iii)*. Finally, *(ii)* follows from *(iii)* by observing that $\mathrm{Ker}\mathbf{G}_N(\mathbf{R}) = 0$ if and only if $K = N$ and $N_k = 1$ for all $k = 1, ..., N$. $\square$

For the particular choice of covariance function defined by Eq.(4) for $0 < \zeta < 2$, we need also the following crucial lower bound:

Proposition 2 *For each $0 < \zeta < 2$ and $d \geq 2$, there exists for each $N \geq 2$ a constant $C_N = C_N(d, \zeta) > 0$ so that the minimum eigenvalue $\lambda_N^{\min}(\mathbf{R})$ of $\mathbf{G}_N(\mathbf{R})$ satisfies*

$$\lambda_N^{\min}(\mathbf{R}) \geq C_N \cdot [\rho(\mathbf{R})]^{\zeta}, \tag{33}$$

with $\rho(\mathbf{R}) = \min_{n \neq m} r_{nm}$.

The above property will be proved in detail in this paper for $N = 2, 3, 4$. While the proof in these cases strongly suggests the result is true for all $N \geq 2$, the argument becomes increasingly complicated for larger values of N. We shall leave the discussion of the general N to a future publication, although we point out that many parts of the argument below apply for the general case. Note that we can view $\mathbf{G}_N$ as a matrix parametrized by the ζ power of the

minimum distance, $\epsilon \equiv \rho^\zeta$. Let $\lambda_N^{\min} = \lambda_N(\epsilon)$ be the minimum positive eigenvalue of $\mathbf{G}_N$ with corresponding unit eigenvector $\Xi_N^{\min} = \Xi_N(\epsilon)$. Then, by the standard formulae of degenerate first-order perturbation theory (see Kato, [14]):

$$\lambda_N(\epsilon) = \langle \Xi_N(0), \mathbf{G}_N(\epsilon)\Xi_N(0)\rangle + O(\epsilon^2). \tag{34}$$

We have used the fact that $\lambda_N(\epsilon)$ is at least twice differentiable in ϵ near zero: see [14], Theorems II.1.8 and II.6.8. Furthermore, $\Xi_N(0)$ is in the null space of $\mathbf{G}_N(0)$. Thus, by Proposition 1*(iii)*, $\Xi_N(0) = (\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_N)$ such that $\sum_{n=1}^N \boldsymbol{\xi}_n = 0$. By simply minimizing over this entire subspace of vectors Ξ, we shall show that the righthand side quadratic form of (34), denoted by $Q_N(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_{N-1})$, is bounded from below by a constant times ϵ. Thus $\lambda(\epsilon)$ obeys the same type of lower bound.

Proposition 2, N=3 Case

Remark: The following proof for Proposition 2, $N = 3$, also implies the lower bound $C_2 r_{12}^\zeta$ for the $N = 2$ case.

Proof: Let $\mathbf{r}_n$, $n = 1, 2, 3$, be three distinct points in $\mathbf{R}^d$, $d \geq 2$. Then we show that there is a positive constant $C_3 = C_3(\rho_0)$, where ρ_0 is the scale of local approximation (11), such that the minimum eigenvalue of $\mathbf{G}_3$ is bounded from below by $C_3\rho^\zeta$. It suffices to treat the situation where $\rho \leq \rho_0$, otherwise, we conclude with Proposition 1. Let C_0 be a large but $O(1)$ constant to be determined, and let $r_{12} = \rho$ for definiteness.

Case I: Suppose now that $\frac{r_{13}}{\rho} \leq C_0$, and $\frac{r_{23}}{\rho} \leq C_0$. By further reducing the size of ρ, we can ensure that $\rho C_0 \leq \rho_0$. Now write:

$$\begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ -\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ 0 \end{pmatrix},$$

then:

$$Q_3 = \langle (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2); \begin{pmatrix} 2(\mathbf{V}(0) - \mathbf{V}_{13}) & \mathbf{V}(0) + \mathbf{V}_{12} - \mathbf{V}_{13} - \mathbf{V}_{23} \\ \mathbf{V}(0) + \mathbf{V}_{12} - \mathbf{V}_{13} - \mathbf{V}_{23} & 2(\mathbf{V}(0) - \mathbf{V}_{23}) \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{pmatrix} \rangle.$$

Since all the three distances are less than ρ_0, we apply lemma 1 to see that $\frac{|\mathbf{V}(0)-\mathbf{V}_{ij}|}{r_{ij}^{\zeta}} \leq C_0$. Therefore we can factor out ρ^{ζ}. The remaining entries are bounded by C_0, and we also know that they form a positive definite matrix. Hence by continuity of eigenvalues on the matrix entries, we get the bound:

$$Q_3 \geq \mu_1(C_0)\rho^{\zeta}, \tag{35}$$

for some positive constant $\mu_1 = \mu_1(C_0)$.

Case II: Suppose $\frac{r_{13}}{\rho} > \frac{C_0}{2}$, $\frac{r_{23}}{\rho} > \frac{C_0}{2}$. By geometric constraint, $\frac{r_{13}}{r_{23}} = 1 + O(C_0^{-1})$. To estimate Q_3 from below, we decompose the vectors $\{(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, -(\boldsymbol{\xi}_1+\boldsymbol{\xi}_2))\}$ into the orthogonal sum of $\{(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0)\}$ and $\{(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -2\boldsymbol{\xi}_1')\}$. Then Q_3 is expressed into the sum of three terms as:

$$\begin{aligned}
Q_3(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) &= \langle(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, -(\boldsymbol{\xi}_1+\boldsymbol{\xi}_2)), \mathbf{G}_3(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, -(\boldsymbol{\xi}_1+\boldsymbol{\xi}_2))^T\rangle \\
&= \langle(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0), \mathbf{G}_3(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0)^T\rangle \\
&+ \langle(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -2\boldsymbol{\xi}_1'), \mathbf{G}_3(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -2\boldsymbol{\xi}_1')^T\rangle \\
&+ 2\langle(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0), \mathbf{G}_3(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -2\boldsymbol{\xi}_1')^T\rangle.
\end{aligned} \tag{36}$$

Write:

$$\begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ -\bar{\boldsymbol{\xi}}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ 0 \\ 0 \end{pmatrix},$$

then the bar term of (36):

$$\begin{aligned}
&\langle(\bar{\boldsymbol{\xi}}_1, 0, 0); \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{V}(0)-\mathbf{V}_{12} & \mathbf{V}_{12} & \mathbf{V}_{13} \\ \mathbf{V}_{12}-\mathbf{V}(0) & \mathbf{V}(0) & \mathbf{V}_{23} \\ \mathbf{V}_{13}-\mathbf{V}_{23} & \mathbf{V}_{23} & \mathbf{V}(0) \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ 0 \\ 0 \end{pmatrix}\rangle \\
&= 2\langle\bar{\boldsymbol{\xi}}_1, (\mathbf{V}(0)-\mathbf{V}_{12})\bar{\boldsymbol{\xi}}_1\rangle \geq \bar{c}_1\rho^{\zeta}|\bar{\boldsymbol{\xi}}_1|^2,
\end{aligned} \tag{37}$$

where $\bar{c}$ here and after will denote a positive constant depending only on ρ_0. Also 1 is a shorthand for $d \times d$ identity matrix. Similarly, we express:

$$\begin{pmatrix} \boldsymbol{\xi}_1' \\ \boldsymbol{\xi}_1' \\ -2\boldsymbol{\xi}_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \\ 0 \end{pmatrix}$$

and write the prime term by Lemma 2 as:

$$\langle (\boldsymbol{\xi}_1', 0, 0), \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{V}(0) + \mathbf{V}_{12} - 2\mathbf{V}_{13} & \mathbf{V}_{12} & \mathbf{V}_{13} \\ \mathbf{V}_{12} + \mathbf{V}(0) - 2\mathbf{V}_{23} & \mathbf{V}(0) & \mathbf{V}_{23} \\ \mathbf{V}_{13} + \mathbf{V}_{23} - 2\mathbf{V}(0) & \mathbf{V}_{23} & \mathbf{V}(0) \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \\ 0 \end{pmatrix} \rangle$$

$$= \langle \boldsymbol{\xi}_1', (6\mathbf{V}(0) + 2\mathbf{V}_{12} - 4\mathbf{V}_{13} - 4\mathbf{V}_{23})\boldsymbol{\xi}_1' \rangle$$

$$= \langle \boldsymbol{\xi}_1', 8(\mathbf{V}(0) - \mathbf{V}_{13})\boldsymbol{\xi}_1' \rangle + \langle \boldsymbol{\xi}_1', (2(\mathbf{V}_{12} - \mathbf{V}(0)) + 4(\mathbf{V}_{13} - \mathbf{V}_{23}))\, \boldsymbol{\xi}_1' \rangle$$

$$\geq \bar{c} r_{13}^{\zeta} |\boldsymbol{\xi}_1'|^2 - \bar{c}(\rho^{\zeta} + r_{12} r_{13}^{\zeta-1}) |\boldsymbol{\xi}_1'|^2$$

$$\geq \bar{c} r_{13}^{\zeta} |\boldsymbol{\xi}_1'|^2 (1 - \bar{c}\left((\rho r_{13}^{-1})^{\zeta} + (\rho r_{13}^{-1}) \right)) \geq \bar{c}_1 r_{13}^{\zeta} |\boldsymbol{\xi}_1'|^2. \tag{38}$$

The mixed term is equal to :

$$\langle (\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0), \begin{pmatrix} \mathbf{V}(0) + \mathbf{V}_{12} - 2\mathbf{V}_{13} & \mathbf{V}_{12} & \mathbf{V}_{13} \\ \mathbf{V}_{12} + \mathbf{V}(0) - 2\mathbf{V}_{23} & \mathbf{V}(0) & \mathbf{V}_{23} \\ \mathbf{V}_{13} + \mathbf{V}_{23} - 2\mathbf{V}(0) & \mathbf{V}_{23} & \mathbf{V}(0) \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \\ 0 \end{pmatrix} \rangle$$

$$= \langle \bar{\boldsymbol{\xi}}_1, (\mathbf{V}(0) + \mathbf{V}_{12} - 2\mathbf{V}_{13})\boldsymbol{\xi}_1' \rangle - \langle \bar{\boldsymbol{\xi}}_1, (\mathbf{V}_{12} + \mathbf{V}(0) - 2\mathbf{V}_{23})\boldsymbol{\xi}_1' \rangle,$$

$$= \langle \bar{\boldsymbol{\xi}}_1, 2(\mathbf{V}_{23} - \mathbf{V}_{13})\boldsymbol{\xi}_1' \rangle,$$

and so is bounded by:

$$|mixed\ \ term| \leq \bar{c}_2 |\boldsymbol{\xi}_1'| \cdot |\bar{\boldsymbol{\xi}}_1| \cdot r_{12} r_{13}^{\zeta-1}. \tag{39}$$

Thus:

$$Q_3 = Q_3(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \geq \bar{c}_1 \rho^{\zeta} |\bar{\boldsymbol{\xi}}_1|^2 + \bar{c}_1 r_{13}^{\zeta} |\boldsymbol{\xi}_1'|^2 - \bar{c}_2 |\boldsymbol{\xi}_1'| \cdot |\bar{\boldsymbol{\xi}}_1| r_{12} r_{13}^{\zeta-1} \tag{40}$$

The mixed term may then be controlled by the positive terms through the following Young's inequality:

$$\begin{aligned}
|\boldsymbol{\xi}_1'|\cdot|\bar{\boldsymbol{\xi}}_1|\rho r_{13}^{\zeta-1} &= \sqrt{\theta\rho^{\zeta}}|\bar{\boldsymbol{\xi}}_1|\cdot\frac{\rho^{1-\frac{\zeta}{2}}r_{13}^{\zeta-1}}{\sqrt{\theta}}|\boldsymbol{\xi}_1'| \\
&\le \frac{1}{2}\theta\cdot\rho^{\zeta}|\bar{\boldsymbol{\xi}}_1|^2+\frac{(\rho/r_{13})^{2-\zeta}}{2\theta}\cdot r_{13}^{\zeta}|\boldsymbol{\xi}_1'|^2, \qquad (41)
\end{aligned}$$

with θ a small number in $(0,1)$. Then, since $\rho/r_{13} < 2C_0^{-1}$, it follows that for any $\zeta < 2$, $(\rho/r_{13})^{2-\zeta} < \theta^2$ for C_0 large enough. Thus,

$$|\boldsymbol{\xi}_1'|\cdot|\bar{\boldsymbol{\xi}}_1|\rho r_{13}^{\zeta-1} \le \frac{1}{2}\theta\cdot\rho^{\zeta}|\bar{\boldsymbol{\xi}}_1|^2+\frac{1}{2}\theta\cdot r_{13}^{\zeta}|\boldsymbol{\xi}_1'|^2, \qquad (42)$$

which allows the mixed term to be absorbed into the positive bar and prime terms. Combining (40-42), we conclude that:

$$Q_3(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\ge\bar{c}\rho^{\zeta}|\bar{\boldsymbol{\xi}}_1|^2+\bar{c}r_{13}^{\zeta}|\boldsymbol{\xi}_1'|^2, \qquad (43)$$

which in the original $(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ variables reads:

$$Q_3(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\ge\bar{c}\rho^{\zeta}|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2|^2+\bar{c}r_{13}^{\zeta}|\boldsymbol{\xi}_1+\boldsymbol{\xi}_2|^2. \qquad (44)$$

We finish the proof with inequality (44) and (35). □

Proposition 2, N=4 Case

We now turn to $N=4$, for which inequality (44) is very helpful. Let $\mathbf{r}_n$, $n=1,2,3,4$, be four distinct points in $\mathbf{R}^d$, $d\ge 2$, and assume that r_{12} is the minimum length ρ. Then we show that there is a positive constant $\bar{c}$ depending only on ρ_0 so that the minimum eigenvalue of $\mathbf{G}_4$ is bounded from below by $\bar{c}\rho^{\zeta}$.

Proof: We order r_3 and r_4 according to the lengths of the three sides intersecting at them. The longest length at r_4 is larger than that at r_3. If they are equal, then the second longest length at r_4 is larger than its counterpart at r_3, and so on. Generically, we are able to order r_3 and r_4 this way. Now r_i, $i=1,2,3,4$, determine a tetrahedra in R^d. Due to geometric constraint, r_{23} and r_{13} are on the same order. So are r_{14} and r_{24}. With no loss of generality, we can assume that $r_{13}=r_{23}=\alpha$, and $r_{14}=r_{24}=\beta$. Let r_{34} be γ, which satisfies the inequalities:

$$\gamma\le\alpha+\beta,\ \beta\le\alpha+\gamma;\alpha\le\beta. \qquad (45)$$

We consider all the possibilities under (45).

Case I. Suppose $2 \geq \frac{\gamma}{\beta} \geq C_2^{-1}$, where $C_2 > 0$ is a large constant to be selected. We have four subcases: I 1.1: $1 \leq \frac{\beta}{\alpha} \leq C_1$ and $1 \leq \frac{\alpha}{\rho} \leq C_0$; I 1.2: $1 \leq \frac{\beta}{\alpha} \leq C_1$ and $\frac{\alpha}{\rho} > C_0$; I 2.1: $\frac{\beta}{\alpha} > C_1$ and $1 \leq \frac{\alpha}{\rho} \leq C_0'$; I 2.2: $\frac{\beta}{\alpha} > C_1$ and $1 \leq \frac{\alpha}{\rho} > C_0'$. Case II: $\frac{\gamma}{\beta} < C_2^{-1}$, which implies with (45) that $1 \leq \frac{\beta}{\alpha} \leq \frac{C_2}{C_2 - 1}$. We have two subcases: II 1.1: $1 \leq \frac{\alpha}{\rho} \leq C_0$ and II 1.2: $1 \leq \frac{\alpha}{\rho} > C_0$

As in the analysis for $N = 3$, we assume that ρ is smaller than ρ_0. The I1.1 is very similar to the first case of $N = 3$, in that all lengths are comparable to each other. Writing

$$(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, -(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix} (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, 0)^T,$$

then:

$$Q_4(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$$

$$\begin{pmatrix} 2(\mathbf{V}(0) - \mathbf{V}_{14}) & \mathbf{V}(0) + \mathbf{V}_{12} - \mathbf{V}_{14} - \mathbf{V}_{24} & \mathbf{V}(0) + \mathbf{V}_{13} - \mathbf{V}_{14} - \mathbf{V}_{24} \\ \mathbf{V}(0) + \mathbf{V}_{12} - \mathbf{V}_{14} - \mathbf{V}_{24} & 2(\mathbf{V}(0) - \mathbf{V}_{24}) & \mathbf{V}(0) + \mathbf{V}_{23} - \mathbf{V}_{24} - \mathbf{V}_{34} \\ \mathbf{V}(0) + \mathbf{V}_{13} - \mathbf{V}_{14} - \mathbf{V}_{24} & \mathbf{V}(0) + \mathbf{V}_{23} - \mathbf{V}_{24} - \mathbf{V}_{34} & 2(\mathbf{V}(0) - \mathbf{V}_{34}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \boldsymbol{\xi}_3 \end{pmatrix}.$$

Using lemma 1 again, we can factor out ρ^ζ with remaining matrix being positive and bounded. We find that there is $\mu = \mu(C_0, C_1, C_2)$ such that:

$$Q_4 \geq \mu \rho^\zeta. \tag{46}$$

Now for I 1.2, we decompose $\{(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, -(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3))\}$ into the orthogonal sum of

$$\{(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0, 0)\}$$

and

$$\{(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_2', -2\boldsymbol{\xi}_1' - \boldsymbol{\xi}_2')\}.$$

Then:

$$Q_4(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) = \langle (\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0, 0), \mathbf{G}_4 (\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0, 0)^T \rangle$$

$$+\langle(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_2', -2\boldsymbol{\xi}_1' - \boldsymbol{\xi}_2'), \mathbf{G}_4(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_2', -2\boldsymbol{\xi}_1' - \boldsymbol{\xi}_2')^T\rangle$$

$$+ 2\langle(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, 0, 0), \mathbf{G}_4(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_2', -2\boldsymbol{\xi}_1' - \boldsymbol{\xi}_2')^T\rangle. \tag{47}$$

Writing:

$$\begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ -\bar{\boldsymbol{\xi}}_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

we see that the bar term is equal to:

$$\langle\bar{\boldsymbol{\xi}}_1, 2(\mathbf{V}(0) - \mathbf{V}_{12})\bar{\boldsymbol{\xi}}_1\rangle \geq \bar{c}\rho^\zeta|\bar{\boldsymbol{\xi}}_1|^2, \tag{48}$$

Writing:

$$\begin{pmatrix} \boldsymbol{\xi}_1' \\ \boldsymbol{\xi}_1' \\ \boldsymbol{\xi}_2' \\ -2\boldsymbol{\xi}_1' - \boldsymbol{\xi}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ \boldsymbol{\xi}_2' \\ 0 \\ 0 \end{pmatrix},$$

the mixed term is equal to:

$$\langle\bar{\boldsymbol{\xi}}_1, -2(\mathbf{V}_{14} - \mathbf{V}_{24})\boldsymbol{\xi}_1'\rangle + \langle\bar{\boldsymbol{\xi}}_1, (\mathbf{V}_{13} - \mathbf{V}_{23} + \mathbf{V}_{24} - \mathbf{V}_{14})\boldsymbol{\xi}_2'\rangle. \tag{49}$$

Similarly the prime term is equal to:

$$\langle(\boldsymbol{\xi}_1', \boldsymbol{\xi}_2'), \begin{pmatrix} 8\mathbf{V}(0) - 8\mathbf{V}_{24} & 2\mathbf{V}_{23} - 2\mathbf{V}_{24} - 2\mathbf{V}_{34} + 2\mathbf{V}(0) \\ \mathbf{V}_{23} - 2\mathbf{V}_{24} - 2\mathbf{V}_{34} + 2\mathbf{V}(0) & 2\mathbf{V}(0) - 2\mathbf{V}_{34} \end{pmatrix} (\boldsymbol{\xi}_1', \boldsymbol{\xi}_2')^T\rangle$$

$$+ \langle(\boldsymbol{\xi}_1', \boldsymbol{\xi}_2'), \begin{pmatrix} 2\mathbf{V}_{12} - 2\mathbf{V}(0) + 4(\mathbf{V}_{24} - \mathbf{V}_{14}) & \mathbf{V}_{13} - \mathbf{V}_{23} + \mathbf{V}_{24} - \mathbf{V}_{14} \\ \mathbf{V}_{13} - \mathbf{V}_{23} + \mathbf{V}_{24} - \mathbf{V}_{14} & 0 \end{pmatrix} (\boldsymbol{\xi}_1', \boldsymbol{\xi}_2')^T\rangle. \tag{50}$$

The first matrix of (50) can be expressed as the product:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{V}(0) & \mathbf{V}_{23} & \mathbf{V}_{24} \\ \mathbf{V}_{23} & \mathbf{V}(0) & \mathbf{V}_{34} \\ \mathbf{V}_{24} & \mathbf{V}_{34} & \mathbf{V}(0) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

hence is positive definite and bounded from below by a positive constant $\mu_1(C_1, C_2)$ times $r_{14}^\zeta |(\boldsymbol{\xi}_1', \boldsymbol{\xi}_2')|^2$. It follows that:

$$\begin{aligned} Q_4(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) &\geq \bar{c}\rho^\zeta |\bar{\boldsymbol{\xi}}_1|^2 + \mu_1(C_1, C_2)(|\boldsymbol{\xi}_1'|^2 + |\boldsymbol{\xi}_2'|^2) r_{14}^\zeta - \bar{c}\rho r_{14}^{\zeta-1} |\bar{\boldsymbol{\xi}}_1|(|\boldsymbol{\xi}_1'| + |\boldsymbol{\xi}_2'|) \\ &\quad - \bar{c}(\rho^\zeta + \rho r_{14}^{\zeta-1})|\boldsymbol{\xi}_1'|^2 - \bar{c}\rho(r_{14}^{\zeta-1} + r_{13}^{\zeta-1})(|\boldsymbol{\xi}_1'| \cdot |\boldsymbol{\xi}_2'|) \\ &\geq \bar{c}\rho^\zeta |\bar{\boldsymbol{\xi}}_1|^2 + \mu_1(C_1, C_2)(|\boldsymbol{\xi}_1'|^2 + |\boldsymbol{\xi}_2'|^2) r_{14}^\zeta \geq \bar{c}\rho^\zeta, \end{aligned} \tag{51}$$

where the mixed term is handled as for $N = 3$ with Young's inequality and C_0 is chosen large enough for given C_1 and C_2.

We now consider I 2.1 and I 2.2. Decompose $\{(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, -(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3)\}$ into the orthogonal sum of $\{(\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2, -(\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2), 0)\}$ and $\{(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -3\boldsymbol{\xi}_1')\}$. Then:

$$\begin{aligned} Q_4(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) &= \langle (\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2, -(\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2), 0), \mathbf{G}_4(\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2, -(\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2), 0)^T \rangle \\ &\quad + \langle ((\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -3\boldsymbol{\xi}_1'), \mathbf{G}_4(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -3\boldsymbol{\xi}_1')^T \rangle \\ &\quad + 2\langle (\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2, -(\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2), 0), \mathbf{G}_4(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -3\boldsymbol{\xi}_1')^T \rangle. \end{aligned} \tag{52}$$

Write:

$$\begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ \bar{\boldsymbol{\xi}}_2 \\ -(\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ \bar{\boldsymbol{\xi}}_2 \\ 0 \\ 0 \end{pmatrix}.$$

Then the bar term is equal to:

$$\langle (\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2), \begin{pmatrix} 2(\mathbf{V}(0) - \mathbf{V}_{13}) & \mathbf{V}(0) + \mathbf{V}_{12} - \mathbf{V}_{13} - \mathbf{V}_{23} \\ \mathbf{V}(0) + \mathbf{V}_{12} - \mathbf{V}_{13} - \mathbf{V}_{23} & 2(\mathbf{V}(0) - \mathbf{V}_{23}) \end{pmatrix} (\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2)^T \rangle,$$

which is larger than:

$$\bar{c}(\rho^\zeta |\bar{\boldsymbol{\xi}}_1 - \bar{\boldsymbol{\xi}}_2|^2 + r_{13}^\zeta |\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2|^2), \tag{53}$$

by applying (44) and the $N = 3$ result. We express:

$$(\boldsymbol{\xi}_1', \boldsymbol{\xi}_1', \boldsymbol{\xi}_1', -3\boldsymbol{\xi}_1')^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and so:

$$\mathbf{G}_4 \begin{pmatrix} \boldsymbol{\xi}_1' \\ \boldsymbol{\xi}_1' \\ \boldsymbol{\xi}_1' \\ -3\boldsymbol{\xi}_1' \end{pmatrix} = \begin{pmatrix} \mathbf{V}(0) + \mathbf{V}_{12} + \mathbf{V}_{13} - 3\mathbf{V}_{14} & \mathbf{V}_{12} & \mathbf{V}_{13} & \mathbf{V}_{14} \\ \mathbf{V}_{12} + \mathbf{V}(0) + \mathbf{V}_{23} - 3\mathbf{V}_{24} & \mathbf{V}(0) & \mathbf{V}_{23} & \mathbf{V}_{24} \\ \mathbf{V}_{13} + \mathbf{V}_{23} + \mathbf{V}(0) - 3\mathbf{V}_{34} & \mathbf{V}_{23} & \mathbf{V}(0) & \mathbf{V}_{34} \\ \mathbf{V}_{14} + \mathbf{V}_{24} + \mathbf{V}_{34} - 3\mathbf{V}(0) & \mathbf{V}_{24} & \mathbf{V}_{34} & \mathbf{V}(0) \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The mixed term is equal to:

$$2(\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2, 0, 0) \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{V}(0) + \mathbf{V}_{12} + \mathbf{V}_{13} - 3\mathbf{V}_{14} & \mathbf{V}_{12} & \mathbf{V}_{13} & \mathbf{V}_{14} \\ \mathbf{V}_{12} + \mathbf{V}(0) + \mathbf{V}_{23} - 3\mathbf{V}_{24} & \mathbf{V}(0) & \mathbf{V}_{23} & \mathbf{V}_{24} \\ \mathbf{V}_{13} + \mathbf{V}_{23} + \mathbf{V}(0) - 3\mathbf{V}_{34} & \mathbf{V}_{23} & \mathbf{V}(0) & \mathbf{V}_{34} \\ \mathbf{V}_{14} + \mathbf{V}_{24} + \mathbf{V}_{34} - 3\mathbf{V}(0) & \mathbf{V}_{24} & \mathbf{V}_{34} & \mathbf{V}(0) \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$= 2(\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_2) \begin{pmatrix} \mathbf{V}_{12} - \mathbf{V}_{23} - 3(\mathbf{V}_{14} - \mathbf{V}_{34}) & \mathbf{V}_{12} - \mathbf{V}_{23} \\ \mathbf{V}_{12} - \mathbf{V}_{13} - 3(\mathbf{V}_{24} - \mathbf{V}_{34}) & \mathbf{V}(0) - \mathbf{V}_{23} \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \end{pmatrix}$$

$$= 2\langle \bar{\boldsymbol{\xi}}_1, (\mathbf{V}_{12} - \mathbf{V}_{23} - 3(\mathbf{V}_{14} - \mathbf{V}_{34}))\boldsymbol{\xi}_1' \rangle + 2\langle \bar{\boldsymbol{\xi}}_2, (\mathbf{V}_{12} - \mathbf{V}_{13} - 3(\mathbf{V}_{24} - \mathbf{V}_{34}))\boldsymbol{\xi}_1' \rangle, \tag{54}$$

which can be written as:

$$= 2\langle \bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2, (\mathbf{V}_{12} - \mathbf{V}_{23} - 3(\mathbf{V}_{14} - \mathbf{V}_{34}))\boldsymbol{\xi}_1' \rangle + 2\langle \bar{\boldsymbol{\xi}}_2, ((\mathbf{V}_{23} - \mathbf{V}_{13}) - 3(\mathbf{V}_{24} - \mathbf{V}_{14}))\boldsymbol{\xi}_1' \rangle. \tag{55}$$

It follows that the mixed term is bounded by:

$$\bar{c}|\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2| \cdot |\boldsymbol{\xi}_1'| r_{13}(\max(r_{12}^{\zeta-1}, r_{13}^{\zeta-1}) + r_{14}^{\zeta-1}\mu(C_2))$$

$$+\bar{c}|\bar{\boldsymbol{\xi}}_2| r_{12} r_{13}^{\zeta-1} |\boldsymbol{\xi}_1'| + \bar{c}|\boldsymbol{\xi}_1'| \cdot |\bar{\boldsymbol{\xi}}_2| r_{12} r_{14}^{\zeta-1}.$$

The prime term is equal to:

$$
\begin{aligned}
& \langle \boldsymbol{\xi}_1', (12\mathbf{V}(0) + 2\mathbf{V}_{12} + 2\mathbf{V}_{13} + 2\mathbf{V}_{23} - 6\mathbf{V}_{14} - 6\mathbf{V}_{24} - 6\mathbf{V}_{34})\boldsymbol{\xi}_1' \rangle \\
= \;& \langle \boldsymbol{\xi}_1', 18(\mathbf{V}(0) - \mathbf{V}_{14})\boldsymbol{\xi}_1' \rangle + \langle \boldsymbol{\xi}_1', (2(\mathbf{V}_{12} - \mathbf{V}(0)) + 2(\mathbf{V}_{13} - \mathbf{V}(0)) \\
+ \;& 2(\mathbf{V}_{23} - \mathbf{V}(0)) + 6(\mathbf{V}_{24} - \mathbf{V}_{34}) - 12(\mathbf{V}_{24} - \mathbf{V}_{14}))\boldsymbol{\xi}_1' \rangle \\
\geq \;& \bar{c} r_{14}^{\zeta} |\boldsymbol{\xi}_1'|^2 - \bar{c}(r_{12}^{\zeta} + r_{13}^{\zeta} + r_{23}^{\zeta})|\boldsymbol{\xi}_1'|^2 - \mu(C_2)(r_{23} r_{24}^{\zeta-1} + r_{12} r_{24}^{\zeta-1})|\boldsymbol{\xi}_1'|^2 \\
= \;& \bar{c} r_{14}^{\zeta} |\boldsymbol{\xi}_1'|^2 (1 - \mu(C_2) C_1^{-\zeta} - \mu(C_2) C_1^{-1}) \geq \bar{c} r_{14}^{\zeta} |\boldsymbol{\xi}_1'|^2,
\end{aligned}
$$

if C_1 is chosen large enough for given C_2. In case of I 2.1, the mixed terms involving $r_{14}^{\zeta-1}$ can be controlled by a Young's inequality as in $N = 3$, using C_1 sufficiently large. The terms $r_{12} r_{13}^{\zeta-1} |\boldsymbol{\xi}_1'| \cdot |\bar{\boldsymbol{\xi}}_2|$ and $r_{13} \max\{r_{12}^{\zeta-1}, r_{13}^{\zeta-1}\} |\boldsymbol{\xi}_1'| \cdot |\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2|$ can be estimated by $(C_0')^p r_{12}^{\zeta} = r_{12}^{\zeta/2} \cdot (C_0')^p r_{12}^{\zeta/2}$, $(p = \max\{\zeta, 1\})$, times the $\boldsymbol{\xi}$ bar or prime factors, then using again Young's inequality, thanks to the relatively large coefficient r_{14}^{ζ} in front of $|\boldsymbol{\xi}_1'|^2$. In other words, we use C_1 being much larger than any chosen C_0'. Observe that $|\bar{\boldsymbol{\xi}}_2|^2 \leq \frac{1}{2}|\bar{\boldsymbol{\xi}}_2 - \bar{\boldsymbol{\xi}}_1|^2 + \frac{1}{2}|\bar{\boldsymbol{\xi}}_2 + \bar{\boldsymbol{\xi}}_1|^2$, so that the mixed terms are again controlled by the prime and bar terms. In case of I 2.2, we make C_0' itself large to control the term $r_{12} r_{13}^{\zeta-1} |\boldsymbol{\xi}_1'| \cdot |\bar{\boldsymbol{\xi}}_2|$. The other terms involving r_{14} are standard and controlled by large C_1. Note that if $\zeta \in (0, 1]$

$$
\begin{aligned}
r_{13} \max\{r_{12}^{\zeta-1}, r_{13}^{\zeta-1}\} &= r_{13}^{\zeta/2} r_{12}^{\zeta/2} \left(\frac{r_{13}}{r_{12}}\right)^{1-\frac{\zeta}{2}} \\
&\leq (C_0')^{1-\frac{\zeta}{2}} r_{13}^{\zeta/2} r_{12}^{\zeta/2}
\end{aligned}
$$

Thus when multiplied to $|\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2| \cdot |\boldsymbol{\xi}_1'|$ it is bounded by

$$
\frac{\theta}{2} r_{13}^{\zeta} |\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2|^2 + \frac{(C_0')^{2-\zeta}}{2\theta} r_{12}^{\zeta} |\boldsymbol{\xi}_1'|^2 \leq \frac{\theta}{2} r_{13}^{\zeta} |\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2|^2 + \frac{\theta}{2} r_{14}^{\zeta} |\boldsymbol{\xi}_1'|^2,
$$

with C_1 much larger than chosen C_0'. If $\zeta \in (1, 2)$, $r_{13} \max\{r_{12}^{\zeta-1}, r_{13}^{\zeta-1}\} = r_{13}^{\zeta}$, and its product with $|\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2| \cdot |\boldsymbol{\xi}_1'|$ is bounded by $\frac{\theta}{2} r_{13}^{\zeta} |\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2|^2 + \frac{r_{13}^{\zeta}}{2\theta} |\boldsymbol{\xi}_1'|^2 \leq \frac{\theta}{2} r_{13}^{\zeta} |\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2|^2 + \frac{\theta}{2} r_{14}^{\zeta} |\boldsymbol{\xi}_1'|^2$, since $C_1^{-\zeta} < \theta^2$ for large C_1. Summarizing the above, we conclude that:

$$
Q_4(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) \geq \bar{c} r_{14}^{\zeta} |\boldsymbol{\xi}_1'|^2 + \bar{c} \rho^{\zeta} |\bar{\boldsymbol{\xi}}_1 - \bar{\boldsymbol{\xi}}_2|^2 + \bar{c} r_{13}^{\zeta} |\bar{\boldsymbol{\xi}}_1 + \bar{\boldsymbol{\xi}}_2|^2, \tag{56}
$$

which, in $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$ variables, is:

$$Q_4(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) \geq \bar{c}\left(\rho^\zeta|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 + \alpha^\zeta|\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 - 2\boldsymbol{\xi}_3|^2 + \beta^\zeta|\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3|^2\right). \tag{57}$$

Finally we consider II. The case II 1.1 is no different from I 1.1. Notice that for II 1.2, we have essentially two separate scales $\beta >> \gamma$, thanks to α and β being on the same scale. Decompose $\{(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, -(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3))\}$ into the orthogonal sum of $\{(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_3, -\bar{\boldsymbol{\xi}}_3)\}$ and $\{(\boldsymbol{\xi}'_1, \boldsymbol{\xi}'_1, -\boldsymbol{\xi}'_1, -\boldsymbol{\xi}'_1)\}$. The bar term is:

$$\langle(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_3, -\bar{\boldsymbol{\xi}}_3), \mathbf{G}_4(\bar{\boldsymbol{\xi}}_1, -\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_3, -\bar{\boldsymbol{\xi}}_3)^T\rangle.$$

By writing:

$$\begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ -\bar{\boldsymbol{\xi}}_1 \\ \bar{\boldsymbol{\xi}}_3 \\ -\bar{\boldsymbol{\xi}}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ 0 \\ \bar{\boldsymbol{\xi}}_3 \\ 0 \end{pmatrix},$$

we simplify the bar term into:

$$\langle(\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\xi}}_3), \begin{pmatrix} 2(\mathbf{V}(0) - \mathbf{V}_{12}) & \mathbf{V}_{13} + \mathbf{V}_{24} - \mathbf{V}_{14} - \mathbf{V}_{23} \\ \mathbf{V}_{13} + \mathbf{V}_{24} - \mathbf{V}_{14} - \mathbf{V}_{23} & 2(\mathbf{V}(0) - \mathbf{V}_{34}) \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ \bar{\boldsymbol{\xi}}_3 \end{pmatrix}\rangle. \tag{58}$$

Then the bar term is bounded as:

$$\begin{aligned}
&= 2\langle\bar{\boldsymbol{\xi}}_1, (\mathbf{V}(0) - \mathbf{V}_{12})\bar{\boldsymbol{\xi}}_1\rangle + 2\langle\bar{\boldsymbol{\xi}}_3, (\mathbf{V}(0) - \mathbf{V}_{34})\bar{\boldsymbol{\xi}}_3\rangle + 2\langle\bar{\boldsymbol{\xi}}_1, (\mathbf{V}_{13} - \mathbf{V}_{23} + \mathbf{V}_{24} - \mathbf{V}_{14})\bar{\boldsymbol{\xi}}_3\rangle \\
&\geq \bar{c}\rho^\zeta|\bar{\boldsymbol{\xi}}_1|^2 + \bar{c}r_{34}^\zeta|\bar{\boldsymbol{\xi}}_3|^2 - \bar{c}|\bar{\boldsymbol{\xi}}_1| \cdot |\bar{\boldsymbol{\xi}}_3\,|\mathbf{V}_{13} - \mathbf{V}_{14} - (\mathbf{V}_{23} - \mathbf{V}_{24})| \\
&\geq \frac{\bar{c}}{2}\rho^\zeta|\bar{\boldsymbol{\xi}}_1|^2 + \frac{\bar{c}}{2}r_{34}^\zeta|\bar{\boldsymbol{\xi}}_3|^2.
\end{aligned} \tag{59}$$

To obtain the last inequality we used lemma 3:

$$\begin{aligned}
|\mathbf{V}_{13} - \mathbf{V}_{14} - (\mathbf{V}_{23} - \mathbf{V}_{24})| &\leq \bar{c}r_{12}r_{34}r_{13}^{\zeta-2} = \bar{c}r_{12}^{\zeta/2}r_{34}^{\zeta/2}\frac{r_{12}^{1-\zeta/2}r_{34}^{1-\zeta/2}}{r_{13}^{1-\zeta/2}r_{13}^{1-\zeta/2}} \\
&\leq \bar{c}r_{12}^{\zeta/2}r_{34}^{\zeta/2}C_1^{-(2-\zeta)/2}(C_2 - 1)^{-(2-\zeta)/2}.
\end{aligned}$$

The last term is small for large C_1, C_2 when $\zeta < 2$. Applying Young's inequality yields the same bound as (59).

Next the prime term is simplified by using:

$$\begin{pmatrix} \boldsymbol{\xi}_1' \\ \boldsymbol{\xi}_1' \\ -\boldsymbol{\xi}_1' \\ -\boldsymbol{\xi}_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The prime term becomes:

$$\begin{aligned} &\langle \boldsymbol{\xi}_1', (2\mathbf{V}(0) + 2\mathbf{V}_{12} - 2\mathbf{V}_{13} - 2\mathbf{V}_{14} - 2\mathbf{V}_{24} - 2\mathbf{V}_{23} + 2\mathbf{V}(0) + 2\mathbf{V}_{34})\, \boldsymbol{\xi}_1' \rangle \\ &\qquad = \langle \boldsymbol{\xi}_1', (8\mathbf{V}(0) - 2\mathbf{V}_{13} - 2\mathbf{V}_{14} - 2\mathbf{V}_{23} - 2\mathbf{V}_{24})\, \boldsymbol{\xi}_1' \rangle \\ &\qquad\quad - \langle \boldsymbol{\xi}_1', (4\mathbf{V}(0) - 2\mathbf{V}_{12} - 2\mathbf{V}_{34})\, \boldsymbol{\xi}_1' \rangle \geq \bar{c} r_{13}^{\zeta} |\boldsymbol{\xi}_1'|^2 \end{aligned} \tag{60}$$

The mixed bar-prime term is:

$$\begin{aligned} &\begin{pmatrix} \bar{\boldsymbol{\xi}}_1 \\ 0 \\ \bar{\boldsymbol{\xi}}_3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{V}(0) + \mathbf{V}_{12} - \mathbf{V}_{13} - \mathbf{V}_{14} & \mathbf{V}_{12} & \mathbf{V}_{13} & \mathbf{V}_{14} \\ \mathbf{V}_{12} + \mathbf{V}(0) - \mathbf{V}_{23} - \mathbf{V}_{24} & \mathbf{V}(0) & \mathbf{V}_{23} & \mathbf{V}_{24} \\ \mathbf{V}_{13} + \mathbf{V}_{23} - \mathbf{V}(0) - \mathbf{V}_{34} & \mathbf{V}_{23} & \mathbf{V}(0) & \mathbf{V}_{34} \\ \mathbf{V}_{14} + \mathbf{V}_{24} - \mathbf{V}_{34} - \mathbf{V}(0) & \mathbf{V}_{24} & \mathbf{V}_{34} & \mathbf{V}(0) \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi}_1' \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &\qquad = \langle \bar{\boldsymbol{\xi}}_1, (\mathbf{V}_{23} - \mathbf{V}_{13} + \mathbf{V}_{24} - \mathbf{V}_{14}) \boldsymbol{\xi}_1' \rangle + \langle \bar{\boldsymbol{\xi}}_3, (\mathbf{V}_{13} + \mathbf{V}_{23} - \mathbf{V}_{14} - \mathbf{V}_{24}) \boldsymbol{\xi}_1' \rangle. \end{aligned}$$

Hence the mixed term is bounded by:

$$\bar{c}(r_{12} r_{13}^{\zeta-1} + r_{12} r_{14}^{\zeta-1}) |\boldsymbol{\xi}_1'| \cdot |\bar{\boldsymbol{\xi}}_1| + \bar{c}(r_{34} r_{14}^{\zeta-1} + r_{34} r_{24}^{\zeta-1}) |\boldsymbol{\xi}_1'| \cdot |\bar{\boldsymbol{\xi}}_3|. \tag{61}$$

All the terms in (61) can be estimated as before with Young's inequality, and we have:

$$Q_4(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) \geq \bar{c} \rho^{\zeta} |\bar{\boldsymbol{\xi}}_1|^2 + \bar{c} r_{34}^{\zeta} |\bar{\boldsymbol{\xi}}_3|^2,$$

which is:

$$Q_4(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) \geq \bar{c} \rho^{\zeta} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 + \gamma^{\zeta} |2\boldsymbol{\xi}_3 - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 + \beta^{\zeta} |\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2|^2. \tag{62}$$

Summarizing all the cases, we finish the proof of the proposition. $\square$

3 Properties of the N-Body Convective Operator

We now define a sesquilinear form $h_N[\Psi_N, \Phi_N]$ for $\Psi_N, \Phi_N \in L^2\left(\Omega^{\otimes N}\right)$, by the expression

$$h_N[\Psi_N, \Phi_N] = \int_{\Omega^{\otimes N}} d\mathbf{R}\ \overline{\boldsymbol{\nabla}_{\mathbf{R}}\Psi_N(\mathbf{R})}\boldsymbol{\cdot}\mathbf{G}_N(\mathbf{R})\boldsymbol{\cdot}\boldsymbol{\nabla}_{\mathbf{R}}\Phi_N(\mathbf{R}). \tag{63}$$

and a quadratic form $h_N[\Psi_N] = h_N[\Psi_N, \Psi_N]$. We take as the form domain

$$\mathbf{D}(h_N) = \{\Psi_N \in L^2\left(\Omega^{\otimes N}\right) : \Psi_N \in C^\infty\left(\Omega^{\otimes N}\right), \mathrm{supp}\Psi_N \subseteq \overline{\Omega^{\otimes N}{}_k} \quad \text{for} \quad \text{some} \quad k,$$
$$\text{and} \quad \Psi_N(\mathbf{R}) = 0 \quad \text{for} \quad \mathbf{R} \in \partial\Omega^{\otimes N}\}. \tag{64}$$

Here we made use of an increasing sequence of open subsets of $\Omega^{\otimes N}$ defined as

$$\Omega^{\otimes N}{}_k = \{\mathbf{R} \in \Omega^{\otimes N} : \rho(\mathbf{R}) > \frac{1}{k}\}. \tag{65}$$

Clearly, this form can be expressed as $h_N[\Psi_N, \Phi_N] = \langle \Psi_N, \mathcal{H}_N\Phi_N\rangle$ where $\mathcal{H}_N$ is the positive, symmetric differential operator

$$\mathcal{H}_N = -\frac{1}{2}\sum_{i,j=1}^{d}\sum_{n,m=1}^{N}\frac{\partial}{\partial x_{in}}\left[V_{ij}(\mathbf{r}_n - \mathbf{r}_m)\frac{\partial}{\partial x_{jm}}\cdot\right] \tag{66}$$

with $\mathbf{D}(\mathcal{H}_N) = \mathbf{D}(h_N)$. Our basic object of interest is the self-adjoint (Friedrichs) extension $\widetilde{\mathcal{H}}_N$ of $\mathcal{H}_N$, which corresponds to the operator with Dirichlet b.c. on $\partial\Omega^{\otimes N}$. Note that it will follow from our discussion below that the same extension $\widetilde{\mathcal{H}}_N$ also arises if one chooses $\mathbf{D}(\mathcal{H}_N) = C_0^\infty\left(\Omega^{\otimes N}\right)$, rather than as above. The main properties of $\widetilde{\mathcal{H}}_N$ follow from those of the form h_N which we now consider.

The basic properties of the form are contained in:

Proposition 3 *The sesquilinear form $h_N[\Psi_N, \Phi_N]$ enjoys the following:*

(i) h_N is a nonnegative, closable form.

(ii) For all $\Psi_N \in \mathbf{D}(h_N)$ and for the same constant C_N in Proposition 2,

$$h_N[\Psi_N] \geq C_N \int_{\Omega^{\otimes N}} d\mathbf{R}\ [\rho(\mathbf{R})]^\zeta |\boldsymbol{\nabla}_{\mathbf{R}}\Psi_N(\mathbf{R})|^2. \tag{67}$$

(iii)For all $\Psi_N \in \mathbf{D}(h_N)$ and for the same constant C_N in Proposition 2,

$$h_N[\Psi_N] \geq C_N \cdot \frac{(d-\gamma)^2}{2}\int_{\Omega^{\otimes N}} d\mathbf{R}\ [\rho(\mathbf{R})]^{-\gamma}|\Psi_N(\mathbf{R})|^2. \tag{68}$$

Proof of Proposition 3. *Ad (i)*: non-negativity is obvious from the definition Eq.(63) and Proposition 1*(i)*. That h_N is closable follows from [14], Theorem VI.1.27 and its Corollary VI.1.28. *Ad (ii)*: This follows directly from the definition Eq.(63) and the variational formula for the minimum eigenvalue of $\mathbf{G}_N(\mathbf{R})$. *Ad (iii)*: For the proof of this inequality, we use the Lemma 2 of Lewis [10]. That lemma states that, given an open domain Λ with smooth boundary, then for any function $g \in H^2(\Lambda)$ such that $\triangle_{\mathbf{R}} g(\mathbf{R}) > 0$ for all $\mathbf{R} \in \Lambda$ and for any function $\varphi \in C_0^\infty(\Lambda)$ (i.e. $= 0$ on $\partial\Lambda$), the inequality holds that

$$\begin{aligned} &\int_\Lambda d\mathbf{R} \mid \triangle_{\mathbf{R}}\, g(\mathbf{R})||\varphi(\mathbf{R})|^2 \\ &\quad \leq 4 \int_\Lambda d\mathbf{R} \mid \triangle_{\mathbf{R}}\, g(\mathbf{R})|^{-1} |\boldsymbol{\nabla}_{\mathbf{R}} g(\mathbf{R})|^2 |\boldsymbol{\nabla}_{\mathbf{R}} \varphi(\mathbf{R})|^2, \end{aligned} \tag{69}$$

This is proved by applying Green's first formula and the Cauchy-Schwartz inequality (see [10]). Let us take for each integer $k \geq 1$ the domain $\Lambda_k = \Omega^{\otimes N}{}_k$ defined as in Eq.(65). If we define $g(\mathbf{R}) = [\rho(\mathbf{R})]^\zeta$, then $g \in H^2(\Lambda_k)$ for each k (and, in fact, $g \in C^\infty(\Lambda_k)$). Furthermore,

$$\triangle_{\mathbf{R}}\, g(\mathbf{R}) = 2\zeta(d-\gamma)[\rho(\mathbf{R})]^{-\gamma} > 0, \tag{70}$$

for $d > \gamma$ (which certainly holds if $\zeta > 0$ and $d \geq 2$) and also

$$|\boldsymbol{\nabla}_{\mathbf{R}} g(\mathbf{R})|^2 = 2\zeta^2 [\rho(\mathbf{R})]^{2\zeta-2}. \tag{71}$$

If $\Psi_N \in \mathbf{D}(h_N)$, then for some k sufficiently large $\Psi_N \in C_0^\infty(\Lambda_k)$, and all the conditions for the inequality (69) are satisfied. Hence, we find by substitution that

$$\begin{aligned} &\int_{\Omega^{\otimes N}} d\mathbf{R}\, [\rho(\mathbf{R})]^\zeta |\boldsymbol{\nabla}_{\mathbf{R}} \Psi_N(\mathbf{R})|^2 \\ &\quad \geq \frac{(d-\gamma)^2}{2} \int_{\Omega^{\otimes N}} d\mathbf{R}\, [\rho(\mathbf{R})]^{-\gamma} |\Psi_N(\mathbf{R})|^2, \end{aligned} \tag{72}$$

whenever $\Psi_N \in \mathbf{D}(h_N)$, for $\zeta > 0$ and $d \geq 2$. If we now use together *(ii)* and inequality (72), we obtain *(iii)*. $\square$

Because of item *(i)* we may now pass to the closed form $\widetilde{h}_N$ (see [14], VI.1.4). Its properties are given in the following Proposition 4:

Proposition 4 *The sesquilinear form $\widetilde{h}_N[\Psi_N, \Phi_N]$ enjoys the following:*

(i) $\widetilde{h}_N$ is a nonnegative, closed form.

(ii) The domain $\mathbf{D}(\widetilde{h}_N)$ consists of the Hilbert space $H_{h_N}\left(\Omega^{\otimes N}\right)$ obtained by completion of $C_0^\infty\left(\Omega^{\otimes N}\right)$ in the inner product

$$\langle \Psi_N, \Phi_N \rangle_{h_N} = \langle \Psi_N, \Phi_N \rangle + h_N[\Psi_N, \Phi_N]. \tag{73}$$

In particular, $H_0^1\left(\Omega^{\otimes N}\right) \subset \mathbf{D}(\widetilde{h}_N)$. Alternatively, $\Psi_N \in \mathbf{D}(\widetilde{h}_N)$ iff $\Psi_N \in L^2\left(\Omega^{\otimes N}\right)$, its 1st distributional derivative satisfies $h_N[\Psi_N] < \infty$, and $\gamma_k\left(\Psi_N|_{{\Omega^{\otimes N}}_k}\right) = 0$ for all integer $k \geq 1$, where γ_k is the trace operator from $H^1\left({\Omega^{\otimes N}}_k\right)$ into $L^2\left(\partial\Omega^{\otimes N} \bigcap {\Omega^{\otimes N}}_k\right)$.

(iii) Both the items (ii) and (iii) of Proposition 3 hold for $\widetilde{h}_N[\Psi_N]$ and for all $\Psi_N \in \mathbf{D}(\widetilde{h}_N)$. Furthermore,

$$h_N[\Psi_N] \geq C_N L^{-\gamma} \cdot \frac{(d-\gamma)^2}{2} \|\Psi_N\|_{L^2}^2 \tag{74}$$

also for all $\Psi_N \in \mathbf{D}(\widetilde{h}_N)$. In particular, $\widetilde{h}_N$ is strictly positive.

Proof of Proposition 4: (i) is immediate.

(ii) We first prove the statement that $H_0^1\left(\Omega^{\otimes N}\right) \subset \mathbf{D}(\widetilde{h}_N)$.

To see this, we remark that $\mathbf{D}(h_N)$ is dense in $H_0^1\left(\Omega^{\otimes N}\right)$ for $d \geq 2$. In fact, it is well-known that in a bounded domain Λ the set of functions $C_0^\infty(\Lambda - \Gamma)$, i.e. functions vanishing on $\Gamma \subset \Lambda$ in addition to Λ^c, is dense in $H_0^l(\Lambda)$ if Γ is a finite union of submanifolds with codimension $k \geq 2l$. This follows from standard density theorems for Sobolev spaces: see Ch.III of Adams [15] or Ch.9 of Maz'ja [16]. The Theorem 3.23 of [15] states that $C_0^\infty(\Lambda - \Gamma)$ is dense in $H_0^l(\Lambda)$ iff Γ is a $(2, l)$-polar set, when $\Lambda = \mathbf{R}^D$. However, the same result is true for any open domain Λ. In fact, repeating Adams' argument, if $C_0^\infty(\Lambda - \Gamma)$ is not dense in $H_0^l(\Lambda)$, then there must be a $u \in H_0^l(\Lambda)$ and an element $T \in H_0^{-l}(\Lambda)$, the Banach dual, so that $T(u) = 1$ but $T(f) = 0$ for all $f \in C_0^\infty(\Lambda - \Gamma)$. However, by [15], Theorem 3.10, this T can be identified with an element of $\mathcal{D}'(\Lambda)$ supported on Γ. Since this can further be canonically identified with an element of $\mathcal{D}'(\mathbf{R}^D)$ supported on Γ, the set Γ cannot be $(2, l)$-polar. The other direction

is even simpler. These arguments go back to [17]. On the other hand, by Theorem 9.2.2 of [16] the set Γ is $(2,l)$-polar iff its lower H^l-capacity vanishes, $\underline{\mathrm{Cap}}\left(\Gamma, H^l\right) = 0$. A convenient sufficient condition for zero H^l-capacity is that the Hausdorff $(D-2l)$-dimensional measure of Γ be finite, $\mathcal{H}^{D-2l}(\Gamma) < \infty$. See Proposition 7.2.3/3 and Theorem 9.4.2 of [16]. (This is essentially just the converse of the Frostman theorem, due originally to Erdös & Gillis [18].) In the case considered, the set Γ is of Hausdorff dimension $D-k$, so that $\mathcal{H}^{D-2l}(\Gamma) < \infty$ for $k \geq 2l$ ($= 0$ for $k > 2l$). Thus, the set Γ has zero H^l-capacity as required. Clearly, $\mathbf{D}(h_N)$ defined in the statement of the Proposition 3 above coincides with $C_0^\infty\left(\Omega^{\otimes N} - \Gamma\right)$, where the set $\Gamma = \{\mathbf{R} \in \Omega^{\otimes N} : \mathbf{r}_n = \mathbf{r}_m, n \neq m\}$ has codimension $= d \geq 2$. Therefore, taking $D = Nd$, $l = 1,\ k = d$ and $\Lambda = \Omega^{\otimes N}$ we obtain the density of $\mathbf{D}(h_N)$ in $H_0^1\left(\Omega^{\otimes N}\right)$, as claimed.

As a consequence, for any $\Psi_N \in H_0^1\left(\Omega^{\otimes N}\right)$ there exists a sequence of elements $\Psi_N^{(m)} \in \mathbf{D}(h_N)$ converging in H^1-norm to Ψ_N. Next, we observe that

$$h_N[\Psi_N] \leq B_N \|\Psi_N\|_{H^1}^2 \tag{75}$$

for some coefficient $B_N > 0$. This may be proved by using the variational principle for the maximum eigenvalue $\lambda_N^{\max}(\mathbf{R})$ of $\mathbf{G}_N(\mathbf{R})$ and then the continuity in $\mathbf{R}$ of $\lambda_N^{\max}(\mathbf{R})$ over the compact set $\overline{\Omega^{\otimes N}}$ to infer $\lambda_N^{\max}(\mathbf{R}) \leq B_N$. This inequality states that the H^1-norm is stronger than the h_N-seminorm. Thus, convergence in H^1 norm of $\Psi_N^{(m)} \in \mathbf{D}(h_N)$ to $\Psi_N \in H_0^1\left(\Omega^{\otimes N}\right)$ implies both that $\Psi_N^{(m)} \to \Psi_N$ in L^2 and also that $h_N[\Psi_N^{(m)} - \Psi_N^{(n)}] \to 0$ as $m, n \to \infty$. Comparing with [14],Section VI.1.3 we see that this means precisely that $\Psi_N \in \mathbf{D}(\widetilde{h}_N)$. Therefore, $H_0^1\left(\Omega^{\otimes N}\right) \subset \mathbf{D}(\widetilde{h}_N)$. This is the first statement of *(ii)*.

Next, we recall from [14], Section VI.1.3 that $\mathbf{D}(\widetilde{h}_N)$ is characterized as the Hilbert space obtained by completion of $\mathbf{D}(h_N)$ in the inner-product (73). Since $\mathbf{D}(h_N) \subset C_0^\infty\left(\Omega^{\otimes N}\right)$, this is certainly contained in the Hilbert space defined in *(ii)* above. However, since we have shown that $H_0^1\left(\Omega^{\otimes N}\right) \subset \mathbf{D}(\widetilde{h}_N)$, the completions of $\mathbf{D}(h_N)$ and $C_0^\infty\left(\Omega^{\otimes N}\right)$ are the same.

Finally, we prove the alternative characterization of $\mathbf{D}(\widetilde{h}_N)$ in *(ii)*. We note by Proposition

3*(ii)* that for each $\Psi_N \in \mathbf{D}(h_N)$ and for each k

$$\|\Psi_N\|_{H^1(\Omega^{\otimes N}{}_k)} \leq k^{\zeta} C_N^{-1} \cdot \|\Psi_N\|_{h_N}. \tag{76}$$

Thus, the H_{h_N}-norm is stronger than the $H^1\left(\Omega^{\otimes N}{}_k\right)$-norm on $\mathbf{D}(h_N)|_{\Omega^{\otimes N}{}_k}$. By definition, for each $\Psi_N \in \mathbf{D}(\widetilde{h}_N)$ there is a sequence $\Psi_N^{(m)} \in \mathbf{D}(h_N)$ converging to Ψ_N in H_{h_N}-norm. This sequence must also then converge to $\Psi_N|_{\Omega^{\otimes N}{}_k}$ in $H^1\left(\Omega^{\otimes N}{}_k\right)$-norm. Passing to the limit in (76), one then obtains its validity for all $\mathbf{D}(\widetilde{h}_N)$. This implies that $\mathbf{D}(\widetilde{h}_N)\Big|_{\Omega^{\otimes N}{}_k} \subset H^1\left(\Omega^{\otimes N}{}_k\right)$ for each integer k. Furthermore, the trace γ_k onto the codimension-1 set $(\partial\Omega^{\otimes N}) \bigcap \Omega^{\otimes N}{}_k$ is continuous from $H^1\left(\Omega^{\otimes N}{}_k\right)$ into $H^{1/2}\left((\partial\Omega^{\otimes N}) \bigcap \Omega^{\otimes N}{}_k\right)$. Since $\Psi_N^{(m)} \in \mathbf{D}(h_N)$, we see that $\gamma_k\left(\Psi_N^{(m)}\Big|_{\Omega^{\otimes N}{}_k}\right) = 0$ and, passing to the limit, $\gamma_k\left(\Psi_N|_{\Omega^{\otimes N}{}_k}\right) = 0$ as an element of $H^{1/2}\left((\partial\Omega^{\otimes N}) \bigcap \Omega^{\otimes N}{}_k\right)$. That is the "only if" part of the characterization. The "if" part is very standard. For each Ψ_N obeying the alternative set of conditions and $k \geq 1$, we may define $\widetilde{\Psi}_N^{(k)}$ by extending the restriction $\Psi_N|_{\Omega^{\otimes N}{}_k}$ again to the whole of $\Omega^{\otimes N}$, defining it to be 0 outside of $\Omega^{\otimes N}{}_k$. Because of the conditions on Ψ_N, the new function $\widetilde{\Psi}_N^{(k)} \in H_0^1\left(\Omega^{\otimes N}\right)$ for each $k \geq 1$. See Theorems 3.16 and 7.55 of [15]. Thus, $\widetilde{\Psi}_N^{(k)} \in \mathbf{D}(\widetilde{h}_N)$ for all $k \geq 1$. However,

$$\|\Psi_N - \widetilde{\Psi}_N^{(k)}\|_{h_N} = \left\|\left(1 - \chi_{\Omega^{\otimes N}{}_k}\right)\Psi_N\right\|_{h_N} \tag{77}$$

where $\chi_{\Omega^{\otimes N}{}_k}$ is the characteristic function of $\Omega^{\otimes N}{}_k$. Because $\|\Psi_N\|_{h_N} < \infty$ by assumption, the righthand side goes to zero by dominated convergence as $k \to \infty$. Thus, we conclude that $\lim_{k\to\infty} \|\Psi_N - \widetilde{\Psi}_N^{(k)}\|_{h_N} = 0$, which implies that $\Psi_N \in \mathbf{D}(\widetilde{h}_N)$.

For *(iii)*: We note that the righthand side of inequalities (67) and (68) in Proposition 3 *(ii)* & *(iii)* are just certain weighted H^1-norms and L^2-norms, respectively, and both of these are bounded by the h_N-norm on $\mathbf{D}(h_N)$. Thus, the argument used to extend inequality (76) from $\mathbf{D}(h_N)$ to $\mathbf{D}(\widetilde{h}_N)$ applies also to extending (67)-(68). Noting that $\rho(\mathbf{R}) \leq \operatorname{diam}\Omega = L$ for all $\mathbf{R} \in \Omega^{\otimes N}$, we derive inequality (74) from (68). □

Remark: The proof does not work for $\Omega = \mathbf{T}^d$, the d-dimensional torus. In that case, inequality (67) of Proposition 3*(ii)* is still valid, where $\rho(\mathbf{R}) = \min_{n\neq m, \mathbf{k}\in\mathbf{Z}^d} |\mathbf{r}_n - \mathbf{r}_m + L\cdot\mathbf{k}|$ has now

period L in each direction as required. Unfortunately, the function $g(\mathbf{R}) = [\rho(\mathbf{R})]^{\zeta}$ does not belong to $H^2\left((\mathbf{T}^d)^{\otimes N}\right)$ away from the set Γ where $\rho(\mathbf{R}) = 0$. It has singularities also on the codimension-1 set Γ' of points where $|\mathbf{r}_n - \mathbf{r}_m| = |\mathbf{r}_n^* - \mathbf{r}_m|$, with $\mathbf{r}_n^*$ a periodic image of $\mathbf{r}_n$. Unless the domain of $\mathbf{D}(h_N)$ is chosen to be $= 0$ on Γ', these singularities would contribute a surface term in the Green's formula, invalidating (68). However, if that condition on $\mathbf{D}(h_N)$ is imposed, then the resulting closed form $\widetilde{h}_N$ has Dirichlet b.c. on Γ', which is unphysical. On the other hand, we expect that these are really just problems with the proof and that the inequality (68) still holds with periodic b.c. Methods used to derive general Hardy-Sobolev inequalities ([16], Ch.2) should apply.

We now exploit the previous results to study the Friedrichs extension $\widetilde{\mathcal{H}}_N$ of $\mathcal{H}_N$. Its existence is provided by the First Representation Theorem of forms ([14], Theorem VI.2.1) which states that there is a unique self-adjoint operator $\widetilde{\mathcal{H}}_N$ whose domain $\mathbf{D}(\widetilde{\mathcal{H}}_N)$ is a core for $\widetilde{h}_N$ and for which $\widetilde{h}_N[\Psi_N, \Phi_N] = \langle \Psi_N, \widetilde{\mathcal{H}}_N \Phi_N \rangle$ for every $\Psi_N \in \mathbf{D}(\widetilde{h}_N)$ and $\Phi_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$. We now discuss the essential properties of this operator that we will need later:

Proposition 5 *The Friedrichs extension $\widetilde{\mathcal{H}}_N$ enjoys the following:*

(i) *$\widetilde{\mathcal{H}}_N$ is strictly positive, with lower bound $\geq C_N L^{-\gamma} \cdot \frac{(d-\gamma)^2}{2}$.*

(ii) *The spectrum of $\widetilde{\mathcal{H}}_N$ is pure point.*

Proof of Proposition 5: Ad (i): (74) and [14], Theorem VI.2.6. *Ad (ii)*: We use the Corollary to Lemma 1 of Lewis [10]. His hypothesis $\mathcal{H}1$ is satisfied by the increasing sequence ${\Omega^{\otimes N}}_k$ for integer $k \geq 1$. His hypothesis $\mathcal{H}2$ is true with $H^m = H^1$ and $c_k = C_N \cdot k^{-\zeta}$ as a consequence of (76). Finally, his third hypothesis holds, with the role of his function $p(x)$ played by $C_N \frac{(d-\gamma)^2}{2} \cdot [\rho(\mathbf{R})]^{-\gamma}$ and $\varepsilon_k = C_N \frac{(d-\gamma)^2}{2} \cdot k^{-\gamma}$, by (68). Lewis' proof exploits the Rellich lemma for the domain ${\Omega^{\otimes N}}_k$ to show that the identity injection $I : H_{h_N}\left(\Omega^{\otimes N}\right) \to L^2\left(\Omega^{\otimes N}\right)$ is compact, by approximating it in norm with compact operators $I_k(\Psi_N) = \widetilde{\Psi}_N^{(k)}$, defined above. The segment property holds for ${\Omega^{\otimes N}}_k$, since its boundary is C^∞ except for a finite number of corners where the two parts of its boundary, $\{\mathbf{R} \in \Omega^{\otimes N} : \rho(\mathbf{R}) = k\}$ and $\partial\Omega^{\otimes N}$, intersect. $\square$

4 Proofs of the Main Theorems

We now prove the main results of the paper, using the properties of $\widetilde{\mathcal{H}}_N$ proved in the preceding section. We start with:

Proof of Theorem 1: By a stationary weak solution of (6) at $\kappa = 0$, we mean a sequence of $\Theta_N^* \in L^2\left(\Omega^{\otimes N}\right)$ indexed by $N \geq 1$, such that, for each $N \geq 1$ and for all $\Phi_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$,

$$\langle \widetilde{\mathcal{H}}_N \Phi_N, \Theta_N^* \rangle = \langle \Phi_N, G_N^* \rangle \tag{78}$$

where for $N \geq 2$

$$G_N^*(\mathbf{R}) = \sum_n \overline{f}(\mathbf{r}_n)\Theta_{N-1}^*(...\widehat{\mathbf{r}_n}...) + \sum_{\text{pairs }\{nm\}} F(\mathbf{r}_n, \mathbf{r}_m)\Theta_{N-2}^*(...\widehat{\mathbf{r}_n}...\widehat{\mathbf{r}_m}...) \tag{79}$$

is the inhomogeneous term of Eq.(6) and $G_1^*(\mathbf{r}_1) = \overline{f}(\mathbf{r}_1)$. Because this quantity for $N > 1$ involves the correlations of lower order, our construction will proceed inductively. We may assume that $G_N^* \in L^2\left(\Omega^{\otimes N}\right)$ (in fact, $G_N^* \in H_0^1\left(\Omega^{\otimes N}\right)$ away from the set Γ). This statement is true for $N = 1$ and, for $N \geq 2$, may be assumed to be true for all $M < N$ if the statement in Theorem 1 is taken as an induction hypothesis. Only the above regularity property of G_N^* will be used in the induction step. Thus, it is enough to show that (78) has a unique solution Θ_N for *any* $G_N \in L^2\left(\Omega^{\otimes N}\right)$ for each $N \geq 1$. That is, we must show that for each N

$$\langle \widetilde{\mathcal{H}}_N \Phi_N, \Theta_N \rangle = \langle \Phi_N, G_N \rangle \tag{80}$$

for all $\Phi_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$ has a unique solution $\Theta_N \in L^2\left(\Omega^{\otimes N}\right)$ for any chosen G_N.

We shall first show that $\Theta_N \in \mathbf{D}(\widetilde{h}_N)$ where Θ_N is any weak solution of (80) with $G_N \in L^2$. To do so, we introduce the *smoothing operators*

$$\mathcal{S}_N^\epsilon = (1 + \epsilon\widetilde{\mathcal{H}}_N)^{-1}, \tag{81}$$

In terms of the resolvent operator $R(z, A) = (A - z)^{-1}$ this may be written as

$$\mathcal{S}_N^\epsilon = \frac{1}{\epsilon} R\left(-\frac{1}{\epsilon}, \widetilde{\mathcal{H}}_N\right) = R(-1, \epsilon\widetilde{\mathcal{H}}_N). \tag{82}$$

These smoothing operators have the following properties: First, they are self-adjoint operators with $\|\mathcal{S}_N^\epsilon\| \le 1$ for all $\epsilon > 0$. Second, because $\widetilde{\mathcal{H}}_N$ is closed and $-\frac{1}{\epsilon}$ is in its resolvent set, it follows from the first equality of (82) that $\mathcal{S}_N^\epsilon : L^2\left(\Omega^{\otimes N}\right) \to \mathbf{D}(\widetilde{\mathcal{H}}_N)$. This exhibits the "smoothing" property of the $\mathcal{S}_N^\epsilon$. Third, $\mathcal{S}_N^\epsilon$ for each $\epsilon > 0$ commutes with $\widetilde{\mathcal{H}}_N$, or, more correctly, $\mathcal{S}_N^\epsilon \widetilde{\mathcal{H}}_N \subset \widetilde{\mathcal{H}}_N \mathcal{S}_N^\epsilon$. Finally, because $\lim_{\epsilon\to 0} \epsilon \widetilde{\mathcal{H}}_N = 0$ in the strong resolvent sense, it follows from the second equality of (82) that

$$\lim_{\epsilon\to 0} \|\mathcal{S}_N^\epsilon \Psi_N - \Psi_N\|_{L^2} = 0 \tag{83}$$

for all $\Psi_N \in L^2\left(\Omega^{\otimes N}\right)$. We now observe that, if Θ_N satisfies (80) for any G_N in L^2, then for any $\Phi_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$,

$$\begin{aligned} \widetilde{h}_N[\Phi_N, \mathcal{S}_N^\epsilon \Theta_N] &= \langle \widetilde{\mathcal{H}}_N \Phi_N, \mathcal{S}_N^\epsilon \Theta_N \rangle \\ &= \langle \mathcal{S}_N^\epsilon \widetilde{\mathcal{H}}_N \Phi_N, \Theta_N \rangle \\ &= \langle \widetilde{\mathcal{H}}_N \mathcal{S}_N^\epsilon \Phi_N, \Theta_N \rangle \\ &= \langle \mathcal{S}_N^\epsilon \Phi_N, G_N \rangle = \langle \Phi_N, \mathcal{S}_N^\epsilon G_N \rangle. \end{aligned} \tag{84}$$

In particular, if we apply this to $\Phi_N = \mathcal{S}_N^\epsilon \Theta_N$, then we find for the quadratic form $\widetilde{h}_N[\mathcal{S}_N^\epsilon \Theta_N] = \langle \mathcal{S}_N^\epsilon \Theta_N, \mathcal{S}_N^\epsilon G_N \rangle$ and, thus,

$$\widetilde{h}_N[\mathcal{S}_N^\epsilon \Theta_N] \le \|\Theta_N\|_{L^2} \cdot \|G_N\|_{L^2} \tag{85}$$

uniformly in $\epsilon > 0$. Since, in addition, the form $\widetilde{h}_N$ is closed and $s - \lim_{\epsilon\to 0} \mathcal{S}_N^\epsilon \Theta_N = \Theta_N$ by Eq.(83), it follows from Theorem VI.1.16 of [14] that $\Theta_N \in \mathbf{D}(\widetilde{h}_N)$.

In that case, for any $\Phi_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$, the equation (80) may be rewritten

$$\widetilde{h}_N[\Phi_N, \Theta_N] = \langle \Phi_N, G_N \rangle. \tag{86}$$

Furthermore, $\mathbf{D}(\widetilde{\mathcal{H}}_N)$ is a core for $\mathbf{D}(\widetilde{h}_N)$ by the First Representation Theorem for forms: see [14], Theorem VI.2.1,item *(ii)*. By the same Theorem VI.2.1, item *(iii)*, it follows from (86) that $\Theta_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$ and that

$$\widetilde{\mathcal{H}}_N \Theta_N = G_N \tag{87}$$

with equality as elements of $L^2\left(\Omega^{\otimes N}\right)$. We observe, since $\widetilde{\mathcal{H}}_N^{-1}$ is bounded, that the equation (87) is equivalent to

$$\Theta_N = \widetilde{\mathcal{H}}_N^{-1} G_N \tag{88}$$

However, for any $G_N \in L^2\left(\Omega^{\otimes N}\right)$ the righthand side of (88) exists, again by boundedness of $\widetilde{\mathcal{H}}_N^{-1}$, and it defines an element $\Theta_N = \widetilde{\mathcal{H}}_N^{-1} G_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$. Thus, the weak solution exists and is unique. $\square$

Proof of Theorem 2 (i): The proof of existence and uniqueness here very closely parallels the previous one, but is even easier. For this reason, we will discuss only a few details. As in the previous case, we may begin by introducing a symmetric sesquilinear form,

$$h_N^{(\kappa_p)}[\Psi_N, \Phi_N] = h_N[\Psi_N, \Phi_N] + \sum_{n=1}^{N} \int_{\Omega^{\otimes N}} d\mathbf{R}\ \overline{(-\triangle_{\mathbf{r}_n})^{p/2}\Psi_N(\mathbf{R})} \cdot (-\triangle_{\mathbf{r}_n})^{p/2}\Phi_N(\mathbf{R}). \tag{89}$$

densely defined on either the same domain as before, $\mathbf{D}\left(h_N^{(\kappa_p)}\right) = \mathbf{D}(h_N)$, or, with identical results, $\mathbf{D}\left(h_N^{(\kappa_p)}\right) = C_0^\infty\left(\Omega^{\otimes N}\right)$. Clearly, this is the same as $h_N^{(\kappa_p)}[\Psi_N, \Phi_N] = \langle \Psi_N, \mathcal{H}_N^{(\kappa_p)}\Phi_N\rangle$, where $\mathcal{H}_N^{(\kappa_p)}$ is the differential operator in Eq.(9) with $\mathbf{D}\left(\mathcal{H}_N^{(\kappa_p)}\right) = \mathbf{D}\left(h_N^{(\kappa_p)}\right)$. We may now consider the self-adjoint (Friedrichs) extensions of these operators, denoted $\widetilde{\mathcal{H}}_N^{(\kappa_p)}$, just as before. We may observe that there is a basic inequality,

$$\widetilde{h}_N^{(\kappa_p)}[\Psi_N] \geq \frac{\kappa_p A_N}{L^{2p}} \|\Psi_N\|_{L^2}^2 \tag{90}$$

with some constant $A_N > 0$, for all $\Psi_N \in \mathbf{D}\left(\widetilde{h}_N^{(\kappa_p)}\right)$. This plays the same role in the present proof as inequality (74) of Proposition 4 *(iii)* did in the previous one. It is proved first for $h_N^{(\kappa_p)}[\Psi_N]$ with $\Psi_N \in \mathbf{D}\left(h_N^{(\kappa_p)}\right)$, by expanding the elements of $\mathbf{D}\left(h_N^{(\kappa_p)}\right)$ in a series of eigenfunctions of the Dirichlet Laplacian $(-\triangle_{\mathbf{R}})_D$, which are complete in $H_0^p\left(\Omega^{\otimes N}\right)$. Then, the result is extended to $\widetilde{h}_N^{(\kappa_p)}[\Psi_N]$ by taking limits. Note that the inequality (90), in particular, implies that the operator $\widetilde{\mathcal{H}}_N^{(\kappa_p)}$ is strictly positive, with lower bound $\geq \kappa_p A_N/L^{2p}$. Therefore, the inverse operator $\left[\widetilde{\mathcal{H}}_N^{(\kappa_p)}\right]^{-1}$ is bounded, as before, and unique weak solutions $\Theta_N^{(\kappa_p)*}$ of the stationary equations are easily constructed with its aid.

A last point which requires some explanation is the regularity $\Theta_N^{(\kappa_p)*} \in H_0^p\left(\Omega^{\otimes N}\right)$ of solutions. In fact, it follows as before that $\Theta_N^{(\kappa_p)*} \in \mathbf{D}\left(\widetilde{h}_N^{(\kappa_p)}\right)$. It therefore suffices to show that $\mathbf{D}\left(\widetilde{h}_N^{(\kappa_p)}\right) \subset H_0^p\left(\Omega^{\otimes N}\right)$. We may identify $\mathbf{D}\left(\widetilde{h}_N^{(\kappa_p)}\right)$ as the completion of the pre-Hilbert space $C_0^\infty\left(\Omega^{\otimes N}\right)$ with the inner product

$$\langle \Psi_N, \Phi_N \rangle_{h_N^{(\kappa_p)}} = \langle \Psi_N, \Phi_N \rangle + h_N^{(\kappa_p)}[\Psi_N, \Phi_N]. \tag{91}$$

See [14], Section VI.1.3. However, we have the elementary inequality

$$\left[\sum_{n=1}^{N} k_n^2\right]^{p/2} \leq C_{N,p}\left[\sum_{n=1}^{N} (k_n^2)^{p/2}\right], \tag{92}$$

with $C_{N,p} = N^{(p-2)/2}$ for $p \geq 2$ and $= 1$ for $1 \leq p \leq 2$. Using then the Parseval's equality for Fourier integrals, it follows that the norm $\|\Psi_N\|_{h_N^{(\kappa_p)}}$ is stronger on $C_0^\infty\left(\Omega^{\otimes N}\right)$ than the Sobolev norm

$$\|\Psi_N\|_{H^p}^2 \equiv \|\Psi_N\|_{L^2}^2 + \|(-\triangle_{\mathbf{R}})^{p/2}\Psi_N\|_{L^2}^2. \tag{93}$$

Since $H_0^p\left(\Omega^{\otimes N}\right)$ is defined to be the completion of $C_0^\infty\left(\Omega^{\otimes N}\right)$ in the norm $\|\cdot\|_{H^p}$, it follows that $\mathbf{D}\left(\widetilde{h}_N^{(\kappa_p)}\right) \subset H_0^p\left(\Omega^{\otimes N}\right)$, as required. □

Proof of Theorem 2 (ii): To construct the weak-L^2 limits of $\Theta_N^{(\kappa_p)*}$ for $\kappa_p \to 0$, the main thing that is required are a priori estimates on the L^2-norms uniform in $\kappa_p > 0$. These are provided as follows. First, we note that $\Theta_N^{(\kappa_p)*} \in \mathbf{D}(\widetilde{h}_N)$ because $\Theta_N^{(\kappa_p)*} \in \mathbf{D}\left(\widetilde{\mathcal{H}}_N^{(\kappa_p)}\right)$ and $\mathbf{D}\left(\widetilde{\mathcal{H}}_N^{(\kappa_p)}\right) \subset H^p\left(\Omega^{\otimes N}\right) \subset \mathbf{D}(\widetilde{h}_N)$ for $p \geq 1$. Thus, we may apply Proposition 4 *(iii)*, inequality (74), to calculate that

$$\begin{aligned}
\|\Theta_N^{(\kappa_p)*}\|_{L^2}^2 &\leq C'_N L^\gamma \widetilde{h}_N\left[\Theta_N^{(\kappa_p)*}\right] \\
&\leq C'_N L^\gamma \widetilde{h}_N^{(\kappa_p)}\left[\Theta_N^{(\kappa_p)*}\right] \\
&= C'_N L^\gamma \langle \Theta_N^{(\kappa_p)*}, G_N^{(\kappa_p)*}\rangle \\
&\leq C'_N L^\gamma \|\Theta_N^{(\kappa_p)*}\|_{L^2}\|G_N^{(\kappa_p)*}\|_{L^2}.
\end{aligned} \tag{94}$$

with $C'_N = [C_N(d-\gamma)^2/2]^{-1}$. In other words,

$$\|\Theta_N^{(\kappa_p)*}\|_{L^2} \leq C'_N L^\gamma \|G_N^{(\kappa_p)*}\|_{L^2}. \tag{95}$$

Using the expression (79) for $G_N^{(\kappa_p)*}$ in terms of the lower-order $\Theta_M^{(\kappa_p)*}$, for $M < N$, it follows that

$$\|G_N^{(\kappa_p)*}\|_{L^2} \leq N \cdot \|\overline{f}\|_{L^2(\Omega)} \|\Theta_{N-1}^{(\kappa_p)*}\|_{L^2} + \frac{N(N-1)}{2} \|F\|_{L^2(\Omega\otimes\Omega)} \|\Theta_{N-2}^{(\kappa_p)*}\|_{L^2}. \tag{96}$$

It is then straightforward to prove inductively from (95) and (96) the main L^2-estimates

$$\|G_N^{(\kappa_p)*}\|_{L^2(\Omega^{\otimes N})} \leq 2 \cdot (2K_N F L^\gamma)^{N-1} \cdot N! \tag{97}$$

and

$$\|\Theta_N^{(\kappa_p)*}\|_{L^2(\Omega^{\otimes N})} \leq (2K_N F L^\gamma)^N \cdot N! \tag{98}$$

where $K_N = \max_{1\leq M \leq N} C'_M$ and $F = \max\{\|\overline{f}\|_{L^2(\Omega)}, \|F\|_{L^2(\Omega\otimes\Omega)}^{1/2}\}$.

A further crucial estimate may be extracted from the preceding discussion. Using the inequality $\widetilde{h}_N\left[\Theta_N^{(\kappa_p)*}\right] \leq \|\Theta_N^{(\kappa_p)*}\|_{L^2} \|G_N^{(\kappa_p)*}\|_{L^2}$ contained in (94) and the L^2 bound on $G_N^{(\kappa_p)*}$, (97), it follows that

$$\widetilde{h}_N\left[\Theta_N^{(\kappa_p)*}\right] \leq \left[(2K_N F L^\gamma)^N \cdot N!\right]^2. \tag{99}$$

In other words, we have a uniform bound for the $\Theta_N^{(\kappa_p)*}$ in the norm of the Hilbert space H_{h_N}:

$$\|\Theta_N^{(\kappa_p)*}\|_{h_N} \leq 2 \cdot (2K_N F L^\gamma)^N \cdot N! \tag{100}$$

This follows by combining estimates (98) and (99).

We now consider the weak-L^2 limits of $\Theta_N^{(\kappa_p)*}$ as $\kappa_p \to 0$. We note first, because of the a priori bound (98) and weak compactness of the unit ball in L^2, that any sequence $\kappa_p^{(n)} \to 0$ contains a subsequence $\kappa_p^{(n')}$ such that $w - \lim_{n'\to\infty} \Theta_N^{(\kappa_p^{(n')})*} = \Theta_N^{(0)*}$ exists, and

$$\|\Theta_N^{(0)*}\|_{L^2(\Omega^{\otimes N})} \leq (2K_N F L^\gamma)^N \cdot N! \tag{101}$$

Furthermore, because of the additional a priori estimate (100), we may extract a further subsequence $\kappa_p^{(n'')}$ which converges weakly in H_{h_N}, and the limit then satisfies

$$\|\Theta_N^{(0)*}\|_{h_N} \leq 2 \cdot (2K_N F L^\gamma)^N \cdot N! \tag{102}$$

We wish to characterize all the possible such weak sequential limits $\Theta_N^{(0)*}$. If the weak limits along subsequences $\kappa_p^{(n')}$ are all identical, then, in fact, the weak limit exists and equals the unique subsequential limit.

We shall show that, in fact, all of the weak subsequential limits coincide with Θ_N^*, the unique weak solution of the zero-diffusivity problem. First of all, we observe that for all $N \geq 1$ and all $\Phi_N \in \mathbf{D}\left(\widetilde{\mathcal{H}}_N^{(\kappa_p)}\right)$

$$\langle\left(\widetilde{\mathcal{H}}_N + \kappa_p \sum_{n=1}^{N} (-\triangle_{\mathbf{r}_n})^p\right)\Phi_N, \Theta_N^{(\kappa_p)*}\rangle = \langle\Phi_N, G_N^{(\kappa_p)*}\rangle, \tag{103}$$

because $\Theta_N^{(\kappa_p)*}$ is a weak solution of the pth-hyperdiffusivity equation. As a consequence of Theorem 2*(i)*, we may take $\mathbf{D}\left(\mathcal{H}_N^{(\kappa_p)}\right) = \mathbf{D}\left(h_N^{(\kappa_p)}\right) = C_0^\infty\left(\Omega^{\otimes N}\right)$. Because $\mathbf{D}(\mathcal{H}_N^{(\kappa_p)}) \subset \mathbf{D}(\widetilde{\mathcal{H}}_N^{(\kappa_p)})$, (103) is therefore true for all $\Phi_N \in C_0^\infty\left(\Omega^{\otimes N}\right)$, independent of the value of $\kappa_p > 0$. Passing to the limit along subsequence $\kappa_p^{(n')}$, we then obtain

$$\langle\widetilde{\mathcal{H}}_N \Phi_N, \Theta_N^{(0)*}\rangle = \langle\Phi_N, G_N^{(0)*}\rangle, \tag{104}$$

for all $\Phi_N \in C_0^\infty\left(\Omega^{\otimes N}\right)$. This is not quite the statement that $\Theta_N^{(0)*}$ is a weak solution of the zero-diffusivity equation, with our definitions. For that to be true it is required that (104) hold for all $\Phi_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$. By the same argument as above, $C_0^\infty\left(\Omega^{\otimes N}\right)$ is a dense subset of $\mathbf{D}(\widetilde{h}_N)$ in the Hilbert space H_{h_N}. Thus, we would like to take the limit in H_{h_N} to obtain (104) for all $\Phi_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$, as required. To do so, however, requires that $\Theta_N^{(0)*} \in \mathbf{D}(\widetilde{h}_N)$, so that we may write

$$\widetilde{h}_N\left[\Phi_N, \Theta_N^{(0)*}\right] = \langle\Phi_N, G_N^{(0)*}\rangle, \tag{105}$$

In this form, the limit may be taken to obtain (104) for all $\Phi_N \in \mathbf{D}(\widetilde{\mathcal{H}}_N)$. Thus, to complete the proof, it is enough to show that $\Theta_N^{(0)*} \in \mathbf{D}(\widetilde{h}_N)$.

To demonstrate the latter regularity of $\Theta_N^{(0)*}$, we shall use the second characterization of $\mathbf{D}(\widetilde{h}_N)$ in Proposition 4*(ii)*. We already have the estimate (102). All that is required in addition is to show that

$$\gamma_k\left(\Theta_N^{(0)*}\Big|_{\Omega^{\otimes N}{}_k}\right) = 0 \tag{106}$$

for all $k \geq 1$. To obtain this, we remark that for each k the identity injection $\iota_k : H_{h_N}\left(\Omega^{\otimes N}{}_k\right) \to H^s\left(\Omega^{\otimes N}{}_k\right)$ is compact for any $s < 1$, because the identity injection from $H_{h_N}\left(\Omega^{\otimes N}{}_k\right)$ to $H^1\left(\Omega^{\otimes N}{}_k\right)$ is bounded by (76) and the identity injection $H^1\left(\Omega^{\otimes N}{}_k\right)$ into $H^s\left(\Omega^{\otimes N}{}_k\right)$ is compact, by the Rellich lemma. We may use the above compact embedding for each fixed k to extract by a diagonal argument a further subsequence $\kappa_p^{(n''')}$ such that

$$\lim_{n'''\to\infty} \left\| \Theta_N^{(\kappa_p^{(n''')})*} - \Theta_N^{(0)*} \right\|_{H^s(\Omega^{\otimes N}{}_k)} = 0 \tag{107}$$

for *all* $k \geq 1$. However, for each k, the trace γ_k is continuous as a map from $H^s\left(\Omega^{\otimes N}{}_k\right)$ into $L^2\left(\partial\Omega^{\otimes N} \bigcap \Omega^{\otimes N}{}_k\right)$ when $s > 1/2$. Furthermore,

$$\gamma_k\left(\Theta_N^{(\kappa_p^{(n''')})*} \Big|_{\Omega^{\otimes N}{}_k} \right) = 0 \tag{108}$$

for all n'''. Thus, passing to the limit, we obtain (106). $\square$

5 Concluding Remarks

We make here just a few remarks on some further results of our analysis and some outstanding problems for future work.

(i) Regularity of the Solutions

The construction above produces solutions $\Theta_N^* \in L^2\left(\Omega^{\otimes N}\right)$ and $\in H_0^1\left(\Omega^{\otimes N}\right)$ away from the singular set Γ. In fact, as was mentioned in the Introduction, it is expected that Θ_N^* are Hölder regular, $\Theta_N^* \in C^\gamma\left(\Omega^{\otimes N}\right)$. Such additional regularity of the solutions of the singular-elliptic equations may follow from Harnack inequalities [19, 20].

(ii) N-Dependence of Spectral Gap and Invariant Measure on Scalar Fields

The Proposition 2 has only been fully proved here for $N \leq 4$. Assuming that it holds for general N, the question of the N-dependence of the constant C_N appearing in its statement has also some importance. As we have seen, the solutions Θ_N^* constructed for $\kappa = 0$ obey an L^2-bound

$$\|\Theta_N^*\|_{L^2(\Omega^{\otimes N})} \leq B^N \cdot N! \tag{109}$$

in which B is proportional to the inverse of $\min_{N\geq 1} C_N$. If C_N is bounded from below uniformly in N, then the above constant $B<\infty$. In that case, the correlation functions Θ_N^* determine a *characteristic functional* via the series

$$\Phi^*(\psi)=\sum_{N=0}^{\infty}\frac{i^N}{N!}\langle\psi^{\otimes N},\Theta_N^*\rangle_{L^2(\Omega^{\otimes N})}, \tag{110}$$

absolutely convergent for $\|\psi\|_{L^2(\Omega)}<B^{-1}$. A measure μ^* on scalar fields $\theta\in L^2(\Omega)$ such that

$$\Phi^*(\psi)=\int_{L^2(\Omega)} e^{i\langle\psi,\theta\rangle}\mu^*(d\theta), \tag{111}$$

is therefore uniquely determined by the correlation functions. That such a measure actually exists is a consequence of the Minlos-Sazonov theorem (see [21], Theorem V.5.1), if it can be shown, for example, that Φ^* defined by Eq.(110) is a positive-definite, weakly continuous functional on $L^2(\Omega)$ and Θ_2^* is the kernel of a trace-class operator on $L^2(\Omega)$, i.e. $\int_\Omega d\mathbf{r}\ \Theta_2^*(\mathbf{r},\mathbf{r})<\infty$. The latter would follow from the regularity discussed in *(i)*.

The measure μ^* so constructed would be the natural candidate for an invariant measure on the scalar fields. Whether the dynamical equation Eq.(1) itself can make sense for $\kappa=0$ with Dirichlet b.c., even in a suitable weak sense, is an unresolved issue. However, for a periodic domain, or $\Omega=\mathbf{T}^d$, the d-dimensional torus, there should be a sensible theory of weak solutions to Eq.(1) and we conjecture that the reconstructed measure μ^* will be invariant under realizations evolving according to that equation.

(iii) Time-Dependent Solutions and Relaxation to the Steady-State

From our work in this paper there follow some further results for time-dependent solutions of the correlation equations, Eq.(6). In fact, by standard semigroup theory (see [14], Ch.IX), a unique solution to Eq.(6) with initial datum $\Theta_N(0)\in L^2\left(\Omega^{\otimes N}\right)$ may be (inductively) obtained via the Riemann integrals

$$\Theta_N(t)=e^{-t\widetilde{\mathcal{H}}_N}\Theta_N(0)+\int_0^t ds\ e^{-(t-s)\widetilde{\mathcal{H}}_N}G_N(s) \tag{112}$$

with G_N given by Eq.(79), in terms of the strongly continuous, contraction semigroups $T_N(t)=e^{-t\widetilde{\mathcal{H}}_N}$. We refrain from precise theorem statements here. Furthermore, because of the strict

positivity of spectrum of $\widetilde{\mathcal{H}}_N$ established here, the semigroup $T_N(t)$ is strictly contractive and the limit exists

$$\lim_{t\to\infty} \|\Theta_N(t) - \Theta_N^*\|_{L^2(\Omega^{\otimes N})} = 0. \tag{113}$$

Thus, the time-dependent solutions converge strongly in L^2 to the stationary solutions constructed in this work. All of the results on existence of solutions for $\kappa_p > 0$ and their convergence to zero-diffusivity solutions for $\kappa_p \to 0$, which were proved above for stationary solutions, also carry over to the time-dependent solutions.

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